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On the Relation between Eulerian and Lagrangian Averages in the Statistical Theory of Turbulence

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Abstract

Two important types of probing of a turbulent velocity field $\vec{u}(\vec{r}, t)$ are the Eulerian probings defined by $d\vec{r}/dt = \vec{v}$ ($\vec{v}$ constant) and the Lagrangian probing defined by $d\vec{r}/dt = \vec{u}(\vec{r}, t)$. In the case of fully developed isotropic and homogeneous turbulence, explicit expressions in terms of the energy spectrum are derived for the autocorrelation coefficients and power spectra obtained by Eulerian and Lagrangian probing. The derivations, which are here given in detail, are based on a statistical representation of the turbulent velocity field using the results of the equilibrium theory of turbulence. The Tailor hypothesis is verified in the limit of high probing velocities. The Hay-Pasquill conjecture relating the Lagrangian and Eulerian power spectra is obtained as an approximation to the transformation equations. Application of the results to the theory of turbulent diffusion is indicated.

Contents

1. Introduction

In the statistical theory of turbulence two different types of velocity correlations with respect to time are of particular interest. They are related to two well-known alternative ways of probing the fluctuating part of the velocity field of the turbulent fluid as a function of time. An Eulerian velocity correlation is obtained by probing in a point which is fixed in a given frame of reference while the Lagrangian velocity correlation is obtained by probing in a point which moves with the fluid particles. In the case of homogeneous and stationary turbulence a particular frame of reference distinguishes itself by being the one in which the average velocity of the fluid is zero; we shall call it the fixed frame of reference. The velocity of the fluid at a point $\vec{r}$ and a time t in the fixed frame of reference constitutes the fluctuating part of the turbulent velocity field at this point and time, and will be denoted by $\vec{u}(\vec{r}, t)$.

An important application of Lagrangian velocity correlations to the problem of turbulent diffusion was made by Taylor (ref. 1) in deriving the relation (1.1) below. This relation describes the diffusion in the fixed frame of reference of a particle moving with the local velocity of the turbulent fluid. The relation gives the time dependence of the mean square particle displacement in the x-direction $\overline{x^2}$, in terms of the variance $\overline{\dot{x}^2} = \overline{u^2}$, of the fluctuating part of the velocity in this direction and a double integral over the Lagrangian autocorrelation coefficient $R_L(\tau)$. The relation has the form

$$\overline{x^2}(t) = 2\overline{u^2} \int_0^t dt' \int_0^{t'} d\tau \, R_L(\tau). \tag{1.1}$$

In the derivation of (1.1), homogeneous and stationary turbulence is assumed, and $R_L(\tau)$ is defined through the equation

$$\overline{\dot{x}^2} \, R_L(\tau) = \overline{\dot{x}(t) \, \dot{x}(t+\tau)} \quad , \tag{1.2}$$

where the averagings are to be carried out as ensemble averagings over Lagrangian probings, i.e. $\dot{x}(t)$ and $\dot{x}(t+\tau)$ are the velocities of the same particle at two different times. Because of the stationariness the average is independent of t, and one may furthermore assume, as it is often done,

that under proper ergodic conditions the averaging might as well be carried out over time for a single realization of the system. In fact, on the assumption of spatial homogeneity, the sample averaging might also be carried out as a space averaging over the initial positions of the particles which are followed in time. The ergodic problems connected with the comparison of results obtained by different averaging procedures are beyond the scope of the present investigation, and we shall use whatever procedure is most convenient.

Experimentally, it is in most cases simpler to measure velocity correlations by Eulerian probing, since it only requires measurement of the fluid velocities as functions of time in fixed points of some convenient frame of reference. Thus it is usually simple to measure the Eulerian autocorrelation coefficient $R_E(\tau)$ defined by

$$\overline{u^2}\, R_E(\tau) = \overline{u(t)\, u(t+\tau)} \quad , \tag{1.3}$$

where the averagings are carried out over Eulerian probings, i. e. $u(t)$ and $u(t+\tau)$ are the fluctuating parts of the fluid velocities at times t and $t+\tau$ respectively, measured at a point fixed in a convenient frame of reference. However, since the frame of reference which is convenient to use for the measurement may not necessarily be the fixed frame of reference mentioned above, this definition is ambiguous, and $R_E(\tau)$ generally depends on the average speed of the fluid relative to the frame of reference used for the measurement.

Thus considerable theoretical and practical interest is attached to the problem of investigating the relations between the different Eulerian autocorrelations and the Lagrangian one. The present investigation attempts to clarify this problem, using a model which seems most suitable in the limiting case of fully developed turbulence. This case represents an asymptotic situation which is often realized to a fair approximation under natural conditions where the distribution of the fluctuating part of the velocity is Gaussian, and where one may assume that the velocity at a particular point in space is the result of a great many largely independent and randomly occurring disturbances. In dealing with these disturbances in the model it has not been necessary to specify their physical origin in detail. However, for many purposes they may be thought of as the velocity fields of randomly distributed eddies or as the velocity contributions from all the vorticity-containing volume elements in the fluid. One particular advantage

of the present model is that it makes it possible to obtain apparently realistic explicit functional relationships between certain of the most commonly used Eulerian and Lagrangian averages.

Although no basic theoretical derivation has hitherto been given of the relation between the Eulerian and Lagrangian correlations, a conjecture supported to some extent by empirical evidence has been put forward by Hay and Pasquill (ref. 2). This conjecture is formulated in terms of Taylor's one-dimensional normalized power spectrum (ref. 3), which is simply the cosine transform of the autocorrelation coefficient and may be defined by

$$P(\omega) = \frac{2}{\pi} \int_0^\infty d\tau \; R(\tau) \cos \omega\tau \quad . \tag{1.4}$$

Thus to each type of autocorrelation coefficient there corresponds a power spectrum. The relation postulated by Hay and Pasquill linking the Eulerian and Lagrangian power spectra is a simple scaling of the frequencies as given by the equation

$$P_L(\omega) = \beta \, P_E(\beta\omega) \quad . \tag{1.5}$$

Empirical evidence as presented by Gifford (ref. 4) indicates that β should be chosen in the range $2 < \beta < 4$. We shall see in section 7 below that the relation (1.5) may be derived as an approximation from more general functional relations expressing P_L and P_E in terms of two different integral transforms of the energy spectum E(k). These more general relations will also permit an interpretation of the parameter β in terms of the ratio, $v/\sqrt{u^2}$, between the mean fluid velocity and the root mean square of the fluctuating part of the velocity field. Furthermore the well-known and - in wind tunnel experiments - often used approximate relation between Eulerian power and energy spectra

$$P_E(\omega) = \frac{1}{v} \, E\left(\frac{\omega}{v}\right) , \tag{1.6}$$

which is occasionally termed Taylor's hypothesis, may be obtained in the limit for $v \gg \sqrt{u^2}$ as an asymptotic form of the more general equation expressing $P_E(\omega)$ as an integral tranform of E(k).

The relations derived in the present work may be applied directly to the problem of turbulent diffusion. The so-called Fickian diffusion leads to the following well-known expression for the mean square displacement $\overline{r^2}$ in the fixed frame of reference:

$$\overline{\vec{r}^2}(t) = D \cdot t \tag{1.7}$$

with D constant. Using the Taylor relation (1.1) and the expression obtained in section 6 below for the Lagrangian autocorrelation coefficient, it is possible to generalize equation (1.7) by introducing a time-dependent diffusion parameter D(t) which may be expressed as an integral transform of the energy spectrum with a time-dependent kernel. From this expression for D(t) it may be determined directly under what conditions and in what time intervals a turbulent velocity field with a given energy spectrum gives rise to either enhanced or Fickian diffusion. Thus the present model covers completely the range of Sutton's diffusion formula (ref. 5), giving at the same time an explicit connection between the diffusion properties and the energy spectrum.

2. Series Expansions of Autocorrelations

For later use we shall recall two important series expansions of the autocorrelation coefficients. The first may be derived directly from the defining equation (1.3) by performing a Taylor expansion of $u(t+\tau)$. It is a well-known feature of this series expansion that, owing to the assumed stationariness of the turbulence, only the even terms survive the averaging process, and the series expansion may then be written as

$$R_E(\tau) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{\overline{u^2}} \overline{\left(\frac{d^n u}{dt^n} \right)_E^2} \frac{\tau^{2n}}{(2n)!} \quad . \tag{2.1}$$

A quite similar expansion may of course be obtained for the Lagrangian autocorrelation coefficient by substituting for u the velocity $\dot{x}(t)$ obtained by Lagrangian probing, and changing the averaging procedure correspondingly.

The other expansion is derivable from a Fourier cosine inversion of equation (1.4) followed by a formal expansion of the cosine, and provided that all moments of ω exist one obtains by performing the integration over each term separately

$$R_E(\tau) = \sum_{n=o}^{\infty} (-1)^n \overline{\omega^{2n}} \frac{\tau^{2n}}{(2n)!} \quad , \tag{2.2}$$

where the averaging is carried out over the Eulerian power spectrum. Also in this case a similar expansion may be obtained for the Lagrangian autocorrelation coefficient by performing the averaging over the Lagrangian power spectrum.

The above notions may easily be extended to space correlations and are in fact conceptually simpler since they involve only averages over simultaneous velocities. Thus the longitudinal space correlation is defined by

$$\overline{u^2} \; f(\xi) = \overline{u(x) \; u(x+\xi)} \quad , \tag{2.3}$$

where the averagings in homogeneous turbulence may be carried out over space. The corresponding normalized one-dimensional longitudinal spectrum is then given by

$$E(k) = \frac{2}{\pi} \int_o^{\infty} d\xi \; f(\xi) \cos k\xi \tag{2.4}$$

while the two series expansions corresponding to equations (2.1) and (2.2) get the analogous forms

$$f(\xi) = \sum_{n=o}^{\infty} (-1)^n \frac{1}{\overline{u^2}} \overline{\left(\frac{d^n u}{dx^n}\right)^2} \frac{\xi^{2n}}{(2n)!} = \sum_{n=o}^{\infty} (-1)^n \overline{k^{2n}} \frac{\xi^{2n}}{(2n)!} \; . \tag{2.5}$$

Equating the coefficients of corresponding terms in the two expansions (2.5), one obtains the relation

$$\overline{\frac{d^n u}{dx^n}^2} = \overline{u^2} \cdot \overline{k^{2n}} \quad , \tag{2.6}$$

which displays the connection between the variance of higher-order space derivatives of the longitudinal velocity and moments of the longitudinal energy spectrum. Similar relations may be obtained for the transverse velocity components.

The relation (2.6) makes it possible to compare the orders of magnitude of the variances of high-order derivatives when one has knowledge of the spectral shape. In well-developed turbulence this shape has been investigated both theoretically and experimentally, and the following general features are well established. Beyond a certain wave number k_1 a region is found, often stretching over several decades in k, where the spectrum may be described by a power law. This region, usually termed the universal equilibrium range, is dominated by the inertial subrange in which theoretical considerations show that the spectrum obeys the Kolmogoroff law

$$E(k) = \text{const. } x\ k^{-5/3} \quad . \tag{2.7}$$

Experimentally this law has gained some support, and in most practical cases of atmospheric turbulence a power law is found with an exponent between -1 and -2 (ref. 10). At even higher wave numbers, viscous effects become dominant, and eventually, beyond a wave number k_2, the spectrum falls off exponentially. Under these circumstances it is possible to show quite generally that for all $n > m > 0$

$$\lim_{\frac{k_1}{k_2} \to 0} \frac{\overline{\left(\frac{d^{n-m}u}{dx^{n-m}}\right)^2 \cdot \left(\frac{d^m u}{dx^m}\right)^2}}{\overline{\left(\frac{d^n u}{dx^n}\right)^2} \cdot \overline{u^2}} = \lim_{\frac{k_1}{k_2} \to 0} \frac{\overline{k^{2(n-m)}} \cdot \overline{k^{2m}}}{\overline{k^{2n}}} = 0 \ . \tag{2.8}$$

In order to illustrate this relation, which will be exploited in later sections, we may calculate the ratio in (2.8) for a rather general spectrum of the type discussed above. Properly normalized in the limit for $k_1 \ll k_2$ it has the form

$$E(k) \simeq \frac{2\ \Gamma(a + \frac{1}{2})}{\sqrt{\pi}\ \ \Gamma(a) k_1} \frac{\exp\left[-(k/k_2)^2\right]}{\left[1 + (k/k_1)^2\right]^{a + \frac{1}{2}}} \ , \tag{2.9}$$

where we shall only be interested in values of a in the interval $0 < a < 1$, corresponding to powers from -1 to -3 for the k-dependence in the inertial subrange of the spectrum. This interval includes the Kolmogoroff value

$-5/3$ corresponding to $\alpha = 1/3$. The function $\Gamma(a) = (a-1)!$ is the usual gamma function as defined for instance in ref. 6. Evaluation of the even moments of k over the spectrum (2.9) gives for $n > 0$ the expression $\overline{k^{2n}} \simeq \left[\Gamma(n-\alpha)\,\Gamma(\alpha+\frac{1}{2})/\sqrt{\pi}\;\Gamma(\alpha)\right](k_1/k_2)^{2\alpha}\,k_2^{2n}$, from which the moment ratio $\epsilon(n, m)$ may be obtained in the same limit for $n > m > 0$:

$$\epsilon(n,m) = \frac{k^{2(n-m)}\cdot k^{2m}}{\overline{k^{2n}}} \simeq \frac{\Gamma(n-m-\alpha)\,\Gamma(m-\alpha)\,\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}\;\Gamma(n-\alpha)\;\Gamma(\alpha)}\left(\frac{k_1}{k_2}\right)^{2\alpha}, \quad (2.10)$$

which is consistent with equation (2.8) above in the indicated range of values for α. Most experiments suggest values of α somewhat larger than $1/3$ and nearer to $1/2$; thus it may usually safely be assumed that the ratio (2.10) is very small for $n > m > 0$ in well-developed turbulence. An estimate of the ratio k_1/k_2 may be obtained in terms of the Reynolds number $R_1 \simeq \sqrt{\overline{u^2}}/\nu k_1$. Using Kolmogoroff's expressions for the energy dissipation rate per unit mass, $\epsilon \simeq k_1(\overline{u^2})^{3/2} \simeq k_2^4\,\nu^3$, one obtains

$$\frac{k_1}{k_2} \simeq R_1^{-3/4} \quad . \quad (2.11)$$

It is thus seen that the ratio (2.10) vanishes at least as $R_1^{-1/2}$ for large Reynolds numbers and $n > m > 0$.

3. The Statistical Model

In formulating the statistical model to be used in the following we have been guided mainly by the statistical theory of shot effect noise as developed by Rice (ref. 7) and to a more modest extent by the dynamic relations governing fluid motion.

In the theory of shot effect noise it is assumed that a random process may be described as a sum of randomly distributed disturbances. It is then possible to express many basic properties of the random process in terms of integrals over single disturbances and their probability of occurrence.

Since the fluctuating part of any velocity component in a turbulent velocity field may be considered a random process, a description in terms of a superposition of a large number of individual disturbances can always be attempted. The physical nature of the disturbances which compose the turbulence and which are used in the statistical model to be presented here

is wide open to discussion. In order to expose more clearly the assumptions and approximations underlying the model we shall, however, make use of a specific physical interpretation of the disturbances constituting the elements of the statistical model. It should nevertheless be made clear at this point that other interpretations than the one presented below might be possible and that the model seems to be sufficiently general to accommodate more than one physical interpretation.

For an approximately incompressible medium the fluctuating part of the velocity field is rotational and can thus in a large space region Ω be written in the form of a volume integral which may be converted into a sum of line integrals as follows:

$$\vec{u}(\vec{r},t) = \int_{\Omega} d^3\vec{R}\ \frac{\vec{\omega} \times (\vec{r}-\vec{R})}{4\pi |\vec{r}-\vec{R}|^3} = \sum_{\text{all } c} \int_c \frac{\Gamma_c}{4\pi} \frac{d\vec{R} \times (\vec{r}-\vec{R})}{|\vec{r}-\vec{R}|^3} , \qquad (3.1)$$

where $d^3\vec{R}$ is the volume element, $\vec{\omega}$ = curl $\vec{u}$ is the local vorticity and Γ_c, defined by $\Gamma_c dR = d^3R$, is the constant vortex strength of a narrow vortex tube, c. The volume integral in (3.1) has been decomposed into a sum over all tubes c, the contribution from each tube being given by a line integral along the tube.

A certain amount of persistence in time and space must be required of the disturbances used in the statistical model. Thus the Helmholtz theorems suggest that a suitable choice for a disturbance might be the velocity field accompanying a section of a vortex tube. Let such a section be characterized by its position in space $\vec{R}_i$, its orientation and size $\Delta\vec{R}_i$ and a constant vortex strength Γ_i; it then contributes to the fluctuating velocity field by a disturbance of the form

$$\vec{F}_i(\vec{r}-\vec{R}_i, \Delta\vec{R}_i) = \frac{\Gamma_i}{4\pi} \frac{\Delta\vec{R}_i \times (\vec{r}-\vec{R}_i)}{|r-R_i|^3} . \qquad (3.2)$$

In order to introduce the statistical element in the theory we assume that the fluctuating part of the turbulent velocity field can be written as a sum

$$\vec{u}(\vec{r},t) = \sum_i \vec{F}_i(\vec{r}-\vec{R}_i, \Delta\vec{R}_i) , \qquad (3.3)$$

where the summation must be extended over all disturbances present, each disturbance representing a few of the many degrees of freedom characterizing a turbulent velocity field. The assumption of homogeneity is satisfied by requiring that at any instant the distribution of disturbance positions $\vec{R}_i$ is random and homogeneous in space. Furthermore isotropy is ensured by requiring the distribution of orientations $\Delta\vec{R}_i$ to be isotropic in space, and finally we shall obtain stationariness of the turbulence by requiring the distributions of vortex strengths Γ_i and sizes $|\Delta\vec{R}_i|$ to be time-independent.

The equation (3.3) is incomplete since the time dependence of $\vec{R}_i$ and $\Delta\vec{R}_i$ has not yet been stated. Equations giving this dependence may, however, be obtained by making use of the fact that according to the Helmholtz theorems, which contain all the dynamics of inviscid flow, vortex lines are also material lines. Provided that the terms giving a significant contribution to (3.3) are disturbances of a size large compared with the scale k_2^{-1} of viscous dissipation, we may thus use equation (3.3) itself in order to obtain the equations of motion for a disturbance, i.e. the position $\vec{R}_i$ and the oriented size $\Delta\vec{R}_i$ of the i'th disturbance must satisfy the equations

$$\dot{\vec{R}}_i = \sum_k \vec{F}_k(\vec{R}_i - \vec{R}_k,\ \Delta\vec{R}_k) \tag{3.4}$$

and

$$\dot{\Delta\vec{R}}_i = \sum_k \left[\vec{F}_k(\vec{R}_i + \Delta\vec{R}_i - \vec{R}_k,\ \Delta\vec{R}_k) - \vec{F}_k(\vec{R}_i - \vec{R}_k,\ \Delta\vec{R}_k) \right] . \tag{3.5}$$

Thus, if the continuous sequence of space points $\vec{r}(t)$ in which the fluctuating part of the velocity field is probed is prescribed in some way as a function of time, then the three equations (3.3), (3.4) and (3.5) completely determine the measured fluctuating velocity $\vec{u}(t)$.

In this investigation we shall make the further important assumption that over the times of interest for the probing process one may neglect the stretching of vortex tubes, i.e. put the right side of equation (3.5) equal to zero, while keeping the time dependence due to the shift of positions of disturbances as indicated in equation (3.4). The present interpretation of the disturbances used in the model thus corresponds to a representation of the velocity field $\vec{u}(\vec{r}, t)$ in terms of a superposition of rigid "eddy" velocity fields, the "eddies" themselves being subject to a motion determined by the compound effect of all the other "eddies". As mentioned above, one is not restricted to this special interpretation of the disturbances F_i so long as they have the properties described in connection with equations (3.3) and (3.4).

We shall now make a few remarks concerning the validity of the approcimations inherent in the statistical model as described above, still using the special physical interpretation of a disturbance. As generally assumed in the equilibrium theory, the turbulence may be characterized by three quantities, the r.m.s. of the fluctuating part of the velocity $\sqrt{\overline{u^2}}$, the scale of the energy-containing eddies $\lambda_1 \simeq k_1^{-1}$ and the Reynolds number $R_1 \simeq \lambda_1 \sqrt{\overline{u^2}}/\nu$. As remarked at the end of section 2, the scale in which viscous effects dominate the motion is approximately given by the expression $\lambda_2 \simeq k_2^{-1} \simeq \lambda_1 R_1^{-3/4}$. The number of degrees of freedom per unit volume in the turbulent motion is estimated to be of the order of $\mu \simeq \lambda_2^{-3} \simeq \lambda_1^{-3} R_1^{9/4}$, which indicates how the complexity of the motion increases with R_1. Since each of the disturbances only contain a few degrees of freedom, the above quantity μ also gives a rough estimate of the spatial density of disturbances.

Thus, if a typical disturbance has a range λ, the number of disturbances contributing to the velocity at a given point in space is of the order of $N \simeq \lambda^3 \mu \simeq (\lambda/\lambda_2)^3$, which, if $\lambda \gg \lambda_2$, implies a large amount of overlap between disturbances and thus justifies the statistical treatment. Another reason for wanting the inequality $\lambda \gg \lambda_2$ to be satisfied is that under these circumstances one is justified in disregarding the effect of viscosity, which is only important for motion in a scale of the order of λ_2.

However, one would also like another inequality, $\lambda_1 \gg \lambda$, to be satisfied for the following reason. Since, according to the theory of Kolmogoroff and Obukhov, the r.m.s. of the velocity fluctuations evaluated over a region of size λ is given by the expression $\sqrt{\overline{u_\lambda^2}} \simeq (\lambda/\lambda_1)^{1/3} \sqrt{\overline{u^2}}$, the validity of the above inequality ensures that the rate of deformation within a region of size λ remains small compared with the mean velocity of displacement of the region. This would tend to justify the neglecting of the effect of stretching of vortex lines.

To see whether the two inequalities may be satisfied simultaneously, a rough estimate of the size of a typical disturbance may be obtained for the spectrum given in equation (2.9) by setting $\lambda^{-2} \approx \overline{k^2}$, from which one obtains, using equation (2.11), the two approximate relations $\lambda \approx \lambda_1 R_1^{-3(1-\alpha)/4}$ and $\lambda \approx \lambda_2 R_1^{3\alpha/4}$. For α in the range $0 < \alpha < 1$ and for large values of the Reynolds number these relations indicate that the two above-mentioned inequalities, $\lambda_1 \gg \lambda \gg \lambda_2$, may be satisfied simultaneously.

In the following sections we shall turn to the problem of comparing averages obtained by Eulerian and Lagrangian probing of the velocity field, using the statistical model described above.

4. The Eulerian-Lagrangian Transformation

As a result of the approximations made in section 3 we may write the equations (3.3) and (3.4) in the simplified form

$$\vec{u}(\vec{r}, t) = \sum_i \vec{F}_i(\vec{r}-\vec{R}_i) \tag{4.1}$$

$$\dot{\vec{R}}_i = \sum_k \vec{F}_k(\vec{R}_i-\vec{R}_k) ,$$

where we have suppressed $\Delta\vec{R}_i$ since it does not enter explicitly in the equations of motion; implicitly, however, it enters in the statistical properties of $\vec{F}_i$. In order to determine $\vec{u}$ as a function of time, the probing path $\vec{r}(t)$ must be specified. This specification is clearly different for Eulerian and Lagrangian probings.

By an Eulerian probing with velocity $\vec{v}$ we shall understand a probing where the point $\vec{r}$ in which the velocity is measured, is moving with a constant velocity $\vec{v}$ with respect to the fixed frame of reference, i.e. the average velocity of the fluid measured in the point $\vec{r}$ is $-\vec{v}$. Thus, with all quantities expressed in the coordinates of the fixed frame of reference, the time dependence of $\vec{r}$ in the Eulerian probing characterized by the constant probing velocity $\vec{v}$ is given by

$$\dot{\vec{r}} = \vec{v} \qquad \text{(Euler)}. \tag{4.2}$$

A special case of the Eulerian probing would be to let $\vec{r}$ remain constant in the fixed frame of reference; this would of course correspond to $\vec{v} = 0$.

In the Lagrangian probing the point r in which the velocity is measured moves with the fluid, and the time dependence of $\vec{r}$ is thus given by the implicit equation

$$\dot{\vec{r}} = \vec{u}(\vec{r}, t) \qquad \text{(Lagrange)} , \tag{4.3}$$

which leads to a considerably more involved motion than equation (4.2).

By solving the equations one would in each of the two cases get $\vec{u}$ as a function of the time t and then in principle be able to calculate the autocorrelations corresponding to Eulerian and Lagrangian probings respectively.

In order to bring forward the essential points without too much complication in notation we shall in this section restrict ourselves to the one-dimensional problem, postponing to a later section the treatment of the general three-dimensional case. Furthermore we shall initially treat the even simpler problem in which the disturbance centres $\vec{R}_i$ are assumed fixed in space, thus reducing the equations (4.1) to a single equation

$$u(x) = \sum_{i=-\infty}^{+\infty} F_i(x-X_i) \quad , \tag{4.4}$$

where the time-independent disturbance position coordinates X_i are assumed to be randomly distributed with a density μ per unit length. The disturbances F_i represent signals with random shapes, which may, however, be classified into a possibly infinite number of types with a corresponding time-independent probability distribution. In order to satisfy the assumption of isotropy it is sufficient in the one-dimensional case to require that the probability distribution of shapes is symmetric, i.e. has the property that shapes $F_i(x)$ and $-F_i(x)$ are equally probable. This also ensures that we are only dealing with the fluctuating part of the velocity since the assumption implies that $\overline{u} = 0$. In this one-dimensional simplified picture, the Eulerian and Lagrangian probings are characterized by the equations

$$\begin{aligned} \dot{x} &= v \qquad \text{(Euler)} \\ \text{and} & \qquad\qquad\qquad\qquad (4.5) \\ \dot{x} &= u(x) \qquad \text{(Lagrange)} \end{aligned}$$

respectively.

Before calculating autocorrelation coefficients corresponding to Eulerian and Lagrangian probings we shall derive a few useful relations which are direct consequences of the simplified statistical model we have just described. They involve moments over the distribution of the fluctuating part of the velocity and its space derivatives.

A general formula has been proved in the appendix which permits us to express expectation values of products of u and its space derivatives, each raised to an arbitrary power p_m and hence having the general form

$$\overline{\prod_{m=0}^{n} \left(d^m u/dx^m\right)^{p_m}} \tag{4.6}$$

in terms of the disturbance density μ and the irreducible mixed moments involving only an average over a single disturbance and its derivatives. The definition of an irreducible mixed moment, which in a space-saving shorthand notation may be written $\langle \prod_{m=0}^{n} [m]^{\pi_m} \rangle$, is

$$\langle \prod_{m=0}^{n} [m]^{\pi_m} \rangle \equiv \langle \int_{-\infty}^{+\infty} dx \prod_{m=0}^{n} (\partial^m F_i / \partial x^m)^{\pi_m} \rangle_i \quad , \tag{4.7}$$

where $\langle \ \rangle_i$ indicates an averaging over disturbance shapes.

With this notation the formula derived in the appendix leads to the expression

$$\prod_{m=0}^{n} (d^m u/dx^m)^{p_m}/p_m ! \ =$$

$$\sum_{s=0}^{s_{max}} \mu^s \Big\{ \sum_{sol\ q} \prod_{\substack{(\pi_o,\ldots,\pi_n) \\ =(0,\ldots,0)}}^{(p_o,\ldots,p_n)} \langle \prod_{m=0}^{n} [m]^{\pi_m}/\pi_m ! \rangle^{q_{\pi_o,\ldots,\pi_n}}/q_{\pi_o,\ldots,\pi_n} ! \Big\} ,$$

which has the form of an expansion in powers of the disturbance density μ. The coefficient to μ^s contains products over all integer number sets $(\pi_o,\ldots,\pi_n)$ satisfying the inequalities $0 \leq \pi_o \leq p_o, \ldots, \ 0 \leq \pi_n \leq p_n$, while the summation should be carried out over all possible different non-negative integer solutions for $q_{\pi_o,\ldots,\pi_n}$ to the Diophantine equations (A.16) and (A.17) subject to the condition (A.19). For large values of μ the dominant term will be the one containing the highest power of μ, i.e. proportional to $\mu^{s_{max}}$. The largest value of s is obtained when the non-vanishing q's have the least possible index sum $\Pi = \sum_{m=0}^{n} \pi_m$. In the present case of symmetric shape distributions the maximum s is obtained when all q's vanish except those for which π is equal to 2, that is, when only pair correlations appear in the corresponding term. The pair correlations may be either quadratic or cross term correlations; in either case the maximum value of s is $s_{max} = p/2$ for a symmetric shape distribution, where p stands for the sum $\sum_{m=o}^{n} p_m$. Another important consequence of the assumption of symmetric

shape distribution is that all moments of odd order vanish; this may of course also be seen from a simple parity argument.

Writing only the dominant term in μ, one obtains for the simple moment $\overline{u^{2n}}$ from equation (4.8)

$$\overline{u^{2n}}/(2n)! \simeq \mu^n \langle [0]^2/2! \rangle^n/n! \quad . \tag{4.9}$$

Substituting in this equation the relation obtained for n = 1, $\overline{u^2} = \mu \langle [0]^2 \rangle$ (which is in this case an exact relation), one obtains

$$\overline{u^{2n}} \simeq \frac{(2n)!}{2^n n!} \left(\overline{u^2}\right)^n \quad . \tag{4.10}$$

Equation (4.10) is characteristic of the moments of a Gaussian distribution and thus ensures that the model, as expected, yields a Gaussian distribution for the velocity u in the limit of large μ, i.e. a high Reynolds number.

Another important special case of equation (4.8) is obtained when one evaluates the variance of the n'th order space derivative of u; in this case the relation is exact and has the form

$$\overline{(d^n u/dx^n)^2} = \mu \langle [n]^2 \rangle \quad . \tag{4.11}$$

Combination of this relation with equation (2.6) yields the following expression for the moments of the energy spectrum:

$$\overline{k^{2n}} = \langle [n]^2 \rangle / \langle [0]^2 \rangle \quad . \tag{4.12}$$

Insertion of this expression for the moments in the ratio (2.10) yields what is later seen to be a useful expansion parameter

$$\epsilon(n,m) = \frac{\langle [n-m]^2 \rangle \langle [m]^2 \rangle}{\langle [n]^2 \rangle \langle [0]^2 \rangle} \qquad (n \rangle m \rangle 0) \quad . \tag{4.13}$$

With these relations at hand one can easily calculate the Eulerian autocorrelation coefficient under the simple assumptions made at the beginning of this section. It may for instance be done by evaluating the terms in the expansion (2.1). With a constant probing velocity v and fixed disturbance centres one obtains

$$\overline{\left(\frac{d^n u}{dt^n}\right)^2_E} = \overline{\left[\left(v\,\frac{d}{dx}\right)^n u\right]^2} = v^{2n}\,\overline{k^{2n}}\;\;\overline{u^2} \quad . \tag{4.14}$$

Inserting this expression in the expansion (2.1) and performing the summation over n, one obtains the following relation between $R_E(\tau)$ and the energy spectrum E(k):

$$R_E(\tau) = \int_0^\infty dk\; E(k)\cos(v\tau k) \quad . \tag{4.15}$$

A comparison of the expansion with equation (2.5) yields the following simple connection between the Eulerian autocorrelation coefficient corresponding to the constant probing velocity v and the one-dimensional space correlation:

$$R_{E,v}(\tau) = f(v\tau) \quad . \tag{4.16}$$

This equation is just the Fourier transform of the relation (1.6); thus Taylor's hypothesis is reproduced as an exact relation in the simplified case under consideration in this section. From equation (4.16) it is realized that the different Eulerian autocorrelations scale in a very simple way with the probing velocity v in this simplified picture. The connection between two such autocorrelation coefficients belonging to the probing velocities v_1 and v_2 respectively is given by

$$R_{E,v_1}(\tau) = R_{E,v_2}\left(\frac{v_1}{v_2}\tau\right); \tag{4.17}$$

the awkward result that for $v_1 = 0$ the autocorrelation coefficient remains 1 for all times is a consequence of the simplifying assumption that the disturbance centres are fixed in space or, as it is frequently formulated, that the turbulence pathern is "frozen". We shall later see that equation (4.17) is only valid as an approximation in the case where both v_1 and v_2 are large compared with $\sqrt{\overline{u^2}}$. In fact, equation (4.16) may be interpreted as saying that in the limit in which one may neglect the motion of the disturbance centres, i.e. for probing velocities $v \gg \sqrt{\overline{u^2}}$, averages obtained by Eulerian probing are equivalent with space averages when properly scaled with the probing velocity v. As another example of this simple scaling law, the relation between the power spectra obtained by Eulerian probing with different velocities is given by

$$P_{E,v_1}(\omega) = \frac{v_2}{v_1} P_{E,v_2}\left(\frac{v_2}{v_1}\omega\right) , \tag{4.18}$$

which is of a form similar to the Hay-Pasquill conjecture (1.5). Although it has not been properly justified here, it has intuitive value to remark that if v_1 is chosen as the r.m.s. of u, i.e. $v_1 = \sqrt{\overline{u^2}}$, and v_2 as the average wind speed, a value of $\beta = v_2/v_1$ would fall within the experimental limits found in nature in many cases and thus give an indication that the Lagrangian probing in some respects resemble an Eulerian probing with a velocity $\sqrt{\overline{u^2}}$.

We may now turn to the less trivial problem of calculating the Lagrangian autocorrelation coefficient $R_L(\tau)$ in the same approximation in which we have just calculated the Eulerian autocorrelation coefficient.

By successive differentiations of the lower one of the two equations (4.5) with respect to time we may obtain the higher-order time derivatives for a Lagrangian probing path in terms of higher-order space derivatives. Differentiating through the space variable x and using the equation which implicitly defines the Lagrangian probing path, one gets for the first few derivatives

$$\left(\frac{d\dot{x}}{dt}\right)_L = \frac{du}{dx} \cdot u$$

$$\left(\frac{d^2\dot{x}}{dt^2}\right)_L = \frac{d^2u}{dx^2} u^2 + \left(\frac{du}{dx}\right)^2 u$$

$$\left(\frac{d^3\dot{x}}{dt^3}\right)_L = \frac{d^3u}{dx^3} u^3 + 4 \frac{d^2u}{dx^2} \frac{du}{dx} u^2 + \left(\frac{du}{dx}\right)^3 u \tag{4.19}$$

$$\left(\frac{d^4\dot{x}}{dt^4}\right)_L = \frac{d^4u}{dx^4} u^4 + 7 \frac{d^3u}{dx^3} \frac{du}{dx} u^3 + 11 \frac{d^2u}{dx^2} \left(\frac{du}{dx}\right)^2 u^2 + 4\left(\frac{d^2u}{dx^2}\right)^2 u^3 + \left(\frac{du}{dx}\right)^4 u ,$$

etc., where the index L indicates that the differentiation with respect to time is carried out for the Lagrangian path.

By induction one may obtain the following explicit expression for the first four terms in the expansion of the n'th derivative with respect to time. These are the only ones containing space derivatives of the order n - 2 or higher:

$$\left(\frac{d^n\dot{x}}{dt^n}\right)_L = \frac{d^nu}{dx^n}\,u^n + B_n\,\frac{d^{n-1}u}{dx^{n-1}}\,\frac{du}{dx}\,u^{n-1} +$$

$$C_n\,\frac{d^{n-2}u}{dx^{n-2}}\,\frac{d^2u}{dx^2}\,u^{n-1} + D_n\,\frac{d^{n-2}u}{dx^{n-2}}\left(\frac{du}{dx}\right)^2 u^{n-2} + \ldots, \quad (4.20)$$

where

$$B_o = B_1 = 0, \quad B_2 = 1 \text{ and } B_n = \frac{n^2-n+2}{2} \quad \text{for } n \geq 3$$

$$C_o = C_1 = C_2 = C_3 = 0, \quad C_4 = 4 \text{ and } C_n = \frac{n^3-3n^2+8n}{6} \quad \text{for } n \geq 5$$

$$D_o = D_1 = D_2 = 0, \quad D_3 = 1 \text{ and } D_n = \frac{3n^4-14n^3+33n^2-46n+48}{24} \quad \text{for } n \geq 4.$$

Squaring equation (4.20) and taking expectation values, one obtains

$$\overline{\left(\frac{d^n\dot{x}}{dt^n}\right)_L^2} = \overline{\left(\frac{d^nu}{dx^n}\right)^2 u^{2n}} + B_n^2\,\overline{\left(\frac{d^{n-1}u}{dx^{n-1}}\right)^2\left(\frac{du}{dx}\right)^2 u^{2n-2}} +$$

$$2C_n\,\overline{\frac{d^nu}{dx^n}\,\frac{d^{n-2}u}{dx^{n-2}}\,\frac{d^2u}{dx^2}\,u^{2n-1}} + \quad (4.21)$$

$$2D_n\,\overline{\frac{d^nu}{dx^n}\,\frac{d^{n-2}u}{dx^{n-2}}\left(\frac{du}{dx}\right)^2 u^{2n-2}} + \ldots,$$

where, for reasons which will be obvious shortly, only those terms have been retained in which the two highest derivatives are either both of even or both of odd order.

All terms in equation (4.21) are of the type for which we have previously stated the general expression (4.8), in terms of the disturbance density μ and the irreducible mixed moments involving only averages over a single disturbance and its derivatives. For instance the first term yields the following contribution to an expansion in terms of the reciprocal disturbance density μ^{-1} :

$$\overline{\left(\frac{d^n u}{dx^n}\right)^2 u^{2n}} = \frac{(2n)!}{2^n n!}\,\mu^{n+1}\,\langle[n]^2\rangle\,\langle[o]^2\rangle^n \left\{\left[1 + 2n\frac{\langle[n][o]\rangle^2}{\langle[n]^2\rangle\langle[o]^2\rangle}\right] + \frac{1}{\mu}\left[n\frac{\langle[n]^2[o]^2\rangle}{\langle[n]^2\rangle\langle[o]^2\rangle} + \frac{n(n-1)}{3!}\frac{\langle[o]^4\rangle}{\langle[o]^2\rangle^2} + \frac{4n(n-1)}{3}\frac{\langle[n][o]^3\rangle\langle[n][o]\rangle}{\langle[n]^2\rangle\langle[o]^2\rangle^2} + \frac{n(n-1)(n-2)}{3}\frac{\langle[o]^4\rangle\langle[n][o]\rangle^2}{\langle[n]^2\rangle\langle[o]^2\rangle^3}\right] + \frac{1}{\mu^2}\left[\ldots\ldots\right] + \ldots\ldots\right\} . \tag{4.22}$$

It should be noted that the leading term, which is proportional to μ^{n+1}, contains only pair correlations while the term which is one power smaller in μ contains also quadruple correlations etc. In the limit for large Reynolds number only the leading term will survive owing to the large value of μ. But even the leading term is composed of two subterms the ratio of which is $2n\langle[n][o]\rangle^2/(\langle[n]^2\rangle\langle[o]^2\rangle)$. However, by partial integrations of the numerator, which contains integrals of the type (4.7), it is seen that this ratio vanishes for odd n and is of the order $2n\,\epsilon(n, \frac{1}{2})$ for even n. Thus only one term survives in the limit of large Reynolds numbers. A similar analysis may be carried through for the second and following terms in equation (4.21). Like the first term they will give rise to terms of the order μ^{n+1} containing pair correlations only and terms of a lower order in μ containing higher-order correlations. However, as it is easily verified, the pair correlation terms will be at least of the order $\epsilon(n, m)$ smaller than the leading term mentioned above and are thus negligible in the limit of large Reynolds numbers. The terms of lower order in μ will also be negligible for the reasons earlier mentioned. Thus the expansion in μ^{-1} and $\epsilon(n, m)$ leads to an expression of the type

$$\overline{\left(\frac{d^n \dot{x}}{dt^n}\right)_L^2} = \overline{\left[\left(u \frac{d}{dx}\right)^n u\right]^2} =$$

$$\frac{(2n)!}{2^n n!} \mu^{n+1} \langle [n]^2 \rangle \langle [o]^2 \rangle^n \left\{ 1 + \sigma(\epsilon) + \right.$$

$$\frac{1}{\mu}\left[n \frac{\langle [n]^2 [o]^2 \rangle}{\langle [n]^2 \rangle \langle [o]^2 \rangle} + \frac{n(n-1)}{3!} \frac{\langle [o]^4 \rangle}{\langle [o]^2 \rangle^2} + \cdots \right] +$$

$$\left. \frac{1}{\mu^2}\left[\cdots \right] + \cdots \right\} \simeq \frac{(2n)!}{2^n n!} \overline{k^{2n}} \left(\overline{u^2}\right)^{n+1} , \quad (4.23)$$

where $\sigma(\epsilon)$ represents terms of at least first order in the small quantities $\epsilon(n, m)$. The last equality is valid in the limit in which the Reynolds number goes to infinity. In deriving the last expression, use has also been made of the equality (4.12).

If the above limit is inserted in the Taylor expansion for the Lagrangian autocorrelation coefficient analogous to (2.1), the following simple result is obtained:

$$R_L(\tau) = \sum_{n=o}^{\infty} \overline{k^{2n}} \left(- \frac{\overline{u^2} \tau^2}{2} \right)^n / n! \quad =$$

$$\int_o^{\infty} dk E(k) \exp\left(- \frac{1}{2} \overline{u^2} \tau^2 k^2 \right) . \quad (4.24)$$

We thus note that in this limit, and neglecting the motion of the disturbance centres, this one-dimensional model makes it possible to express the Lagrangian autocorrelation coefficient in terms of an integral transformation of the one-dimensional energy spectrum with a Gaussian kernel, whereas the Eulerian autocorrelation coefficient is obtained by performing a cosine transformation of the energy spectrum as given by (4.15).

5. Effect of the Motion of Disturbance Centres

In this section we shall return to the full set of equations (4.1), including the motion of the disturbance centres; we shall, however, still restrict the considerations to the one-dimensional problem, which may then be formulated through the equations

$$u(x,t) = \sum_{i=-\infty}^{+\infty} F_i(x-X_i)$$

and (5.1)

$$\dot{X}_i = \sum_{k=-\infty}^{+\infty} F_k(X_i-X_k) ,$$

the last equation standing for the whole set of equations describing the motion of the disturbance centres X_i. The velocity field u(x, t) is now time dependent through the motion of the X_i's.

As before, we define the two different ways of probing the velocity field by the equations

$$\dot{x} = v \qquad \text{(Euler)}$$

and (5.2)

$$\dot{x} = u(x,t) \qquad \text{(Lagrange)} .$$

Taking the variance of the n'th order time derivative of the fluctuating part of the velocity in the Eulerian description, we obtain instead of equation (4.14) the equation

$$\overline{\left(\frac{d^n u}{dt^n}\right)_E^2} \simeq \overline{(v-\dot{X}_i)^{2n}\left(\frac{\partial^n u}{\partial x^n}\right)^2} \simeq \overline{(v-u)^{2n}} \cdot \overline{\left(\frac{\partial^n u}{\partial x^n}\right)^2} , \qquad (5.3)$$

where only the term which becomes dominant after the averaging process has been retained. Also the last equation is only valid in the limit of large Reynolds numbers, where expansions of the type (4.22) may be used. This essentially amounts to saying that the dominant term of the variance (5.3) may be obtained by treating $(\partial^n u/\partial x^n)^2$ as uncorrelated with $(v-\dot{X}_i)^{2n}$. Furthermore, since the statistical variables $\dot{X}_i$ and u are similarly distributed with a Gaussian distribution function in the limit of large μ, as may be

proved by the method described in detail in section 4, one may in a simple way obtain the value of the average $\overline{(v-X_i^*)^{2n}} = \overline{(v-u)^{2n}}$, using the method of characteristic functions (ch. f.) described in ref. 7.

The ch. f. corresponding to the Gaussian distribution of u is given by

$$\varphi_u(\varkappa) = \overline{\exp[i\varkappa u]} = \exp(-\frac{1}{2}\overline{u^2}\varkappa^2) \tag{5.4}$$

while the δ-function distribution of the constant velocity v has the ch. f.

$$\varphi_v(\varkappa) = \overline{\exp[i\varkappa v]} = \exp(i\varkappa v) \ . \tag{5.5}$$

The ch. f. corresponding to the difference v-u is then given by

$$\varphi_{v-u}(\varkappa) = \exp(i\varkappa v - \frac{1}{2}\overline{u^2}\varkappa^2) \ . \tag{5.6}$$

The moments of v-u may now be obtained in the usual way by differentiating the ch. f. (5.6) with respect to $\varkappa$. Using the property (5.7) of the generating function for Hermite polynomials (ref. 6)

$$H_m(y) = \left[\frac{\partial^m}{\partial z^m}\exp(2yz-z^2)\right]_{z=0} \ , \tag{5.7}$$

one obtains the result

$$\overline{(v-u)^{2n}} = (-1)^n\left(\frac{\overline{u^2}}{2}\right)^n H_{2n}\left(\frac{iv}{\sqrt{2\overline{u^2}}}\right) . \tag{5.8}$$

Inserting the result (5.8) in equation (5.3) together with the expression $\overline{(\partial^n u/\partial x^n)^2} = \overline{k^{2n}} \cdot \overline{u^2}$ derivable from equations (4.11) and (4.12), one obtains

$$\frac{1}{\overline{u^2}}\left(\frac{d^n u}{dt^n}\right)_E^2 \simeq \overline{k^{2n}}\,(-1)^n\left(\frac{\overline{u^2}}{2}\right)^n H_{2n}\left(\frac{iv}{\sqrt{2\overline{u^2}}}\right) \ , \tag{5.9}$$

which, inserted in the expansion (2.1) for the Eulerian autocorrelation coefficient, yields the more general relation

$$R_E(\tau) = \int_0^\infty dk\ E(k) \exp(- \frac{1}{2}\ \overline{u^2}\ k^2\tau^2) \cos(vk\tau) \quad , \tag{5.10}$$

where use has been made of the following relation applying to Hermite polynomials of even order:

$$\sum_{m=0}^{\infty} (-1)^m\ H_{2m}(y)z^{2m}/(2m)! \ = \exp(z^2) \cdot \cos(2yz) \quad . \tag{5.11}$$

This expression for $R_E(\tau)$ contains both the constant probing velocity v and the variance $\overline{u^2}$ of the fluctuating part of the velocity as parameters. It has the expected property of being reduced to the usual cosine transform (4.15) in the limit for $\overline{u^2} \ll v^2$. In the opposite limit, $v^2 \ll \overline{u^2}$, it is interesting to note that one obtains the relation (4.24) derived in the previous section for the Lagrangian autocorrelation coefficient when neglecting the motion of the disturbance centres.

Turning now to the Lagrangian case, one may analogously write the variance of the n'th derivative of the fluid particle velocity $\dot{x}$ with respect to time in the form

$$\overline{\left(\frac{d^n\dot{x}}{dt^n}\right)^2_L} \simeq \overline{(u-\dot{X}_i)^{2n}} \cdot \overline{\left(\frac{\partial^n u}{\partial x^n}\right)^2} \quad , \tag{5.12}$$

where, as before, only the dominant term has been retained.

The characteristic function for the difference between the two variables u and $\dot{X}_i$ which have equal Gaussian distributions is given by

$$\varphi_{u-\dot{X}_i} = \exp(- \overline{u^2}\varkappa^2) \quad , \tag{5.13}$$

and the moment appearing in (5.12) then has the value

$$\overline{(u-\dot{X}_i)^{2n}} = (-i)^{2n} \left[\frac{\partial^{2n}}{\partial\varkappa^{2n}}\ \varphi_{u-\dot{X}_i}\right]_{\varkappa=0} = \frac{(2n)!}{n!}\ \overline{(u^2)}^n \quad . \tag{5.14}$$

Making the usual substitutions, one obtains for the Lagrangian autocorrelation the expression

$$R_L(\tau) = \int_0^\infty dk\, E(k) \exp(-\overline{u^2} k^2 \tau^2) \quad , \tag{5.15}$$

which only deviates from the earlier derived expression (4.24) by having $2\overline{u^2}$ instead of $\overline{u^2}$. This is thus the only effect of including the motion of the disturbance centres in the Lagrangian description.

An interesting aspect of the probing problem is connected with the application of weather balloons, where the motion of the probe is neither a translation with constant velocity v nor can be considered a Lagrangian probing since, owing to the finite size of the balloon, it may not be subject to the small-scale motion in the atmosphere. However, if the motion of the balloon in the fixed coordinate system is assumed to be random with a Gaussian distribution with dispersion $\overline{u^2}_B$, a trivial extension of the arguments leading to equation (5.15) gives the relation between the energy spectrum, E(k), and the measured autocorrelation, which we may call $R_B(\tau)$, where B stands for balloon. The relation is

$$R_B(\tau) = \int_0^\infty dk\, E(k) \exp\left(- \frac{\overline{u^2} + \overline{u_B^2}}{2} k^2 \tau^2 \right) \quad . \tag{5.16}$$

All the expressions derived in this section are restricted to the one-dimensional problem. In the following section the treatment will be extended to the three-dimensional case.

6. Extension to Three Dimensions

In order to extend our treatment of the Eulerian-Lagrangian transformation problem to the more realistic three-dimensional problem we shall for a moment return to Taylor's diffusion formula (1.1). Similar relations may be written for the two other space dimensions and in isotropic turbulence $R_L(\tau)$ will be identically the same function in all three equations. Adding the three equations, we obtain

$$\overline{\vec{r}^2}(t) = 2\overline{\vec{u}^2} \int_0^t dt' \int_0^{t'} d\tau\, R_L(\tau) \quad , \tag{6.1}$$

where $R_L(\tau)$ is defined by equation (1.2) or the equivalent equation (6.2):

$$\overline{(\dot{\vec{r}})^2}\ R_L(\tau) = \overline{\dot{\vec{r}}(t) \cdot \dot{\vec{r}}(t+\tau)} \quad , \tag{6.2}$$

where the averaging is carried out over the scalar product of the velocities of the same particle at two instants separated by a time interval τ. We may also recall that the velocity variance is given by $\overline{(\dot{\vec{r}})^2} = \overline{\vec{u}^2} = \overline{u^2}$ where $u = |\vec{u}|$ now denotes the size of the fluctuating part of the three-dimensional velocity vector.

When considering the Eulerian autocorrelation coefficient $R_E(\tau)$, which we want to compare with $R_L(\tau)$, we must define it in a way analogous to (6.2) as

$$\overline{\vec{u}^2}\ R_E(\tau) = \overline{\vec{u}(t) \cdot \vec{u}(t+\tau)} \quad , \tag{6.3}$$

where the averaging is carried out over a probing path characterized by the equation $\dot{\vec{r}} = \vec{v}$.

If we split the scalar product (6.3) into its three components, choosing a coordinate system where $\vec{v}$ is parallel to one of the axes, we may write the autocorrelation coefficient as the following combination of longitudinal and transverse autocorrelations:

$$R_E(\tau) = \frac{1}{3} R_{//}(\tau) + \frac{2}{3} R_{\perp}(\tau) \quad , \tag{6.4}$$

where in the simple case in which the motion of the disturbance centres is neglected, i. e. for large velocities v, the functions $R_{//}$ and $R_{\perp}$ are related to the conventional longitudinal and transverse space correlations coefficients f and g (as defined for instance in ref. 11) by the simple scaling equations

$$R_{//}(\tau) = f(v\tau) \quad \text{and} \quad R_{\perp}(\tau) = g(v\tau) \quad . \tag{6.5}$$

By performing a Taylor expansion of (6.3) and assuming stationariness, we obtain the three-dimensional analogue of equation (2.1)

$$R_E(\tau) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{\vec{u}^2} \left(\frac{d^n \vec{u}}{dt^n} \right)^2_E \frac{\tau^{2n}}{(2n)!} \quad , \tag{6.6}$$

and we shall as before be concerned with the evaluation of the variance of $d^n\vec{u}/dt^n$ appearing in (6.6).

Starting with the simple case of Eulerian averages, where the motion of the disturbance centres may be neglected, we obtain the relation

$$\left(\frac{d^n\vec{u}}{dt^n}\right)_E = (\vec{v}\cdot\vec{\nabla})^n\vec{u} = |\vec{v}|^n(\vec{e}\cdot\vec{\nabla})^n\vec{u}(\vec{r}) \quad , \tag{6.7}$$

where $\vec{e}$ is a unit vector in the direction of $\vec{v}$ and $\vec{e}\cdot\vec{\nabla}$ is the differential operator in the $\vec{e}$ direction. Squaring and taking averages, we obtain the result

$$\frac{1}{u^2}\overline{\left(\frac{d^n\vec{u}}{dt^n}\right)^2}_E = v^{2n}\ \frac{1}{u^2}\ \overline{\left[(\vec{e}\cdot\vec{\nabla})^n\vec{u}\right]^2} = v^{2n}\ \frac{1}{u^2}\ \overline{\left(\frac{\partial^n\vec{u}}{\partial x^n}\right)^2}, \tag{6.8}$$

where, in evaluating the scalar product, the x-axis of the coordinate system has been chosen parallel to $\vec{e}$; the value of the average is, however, of course independent of this particular choice. The scalar quantity $\overline{(\partial^n\vec{u}/\partial x^n)^2}$, which obviously is an intrinsic property of the velocity field and thus is independent of the probing velocity $\vec{v}$, may be obtained in terms of moments of a scalar wave number k over a suitably defined energy spectrum E(k). Choosing E(k) as

$$E(k) = \frac{1}{3}E_{\parallel}(k) + \frac{2}{3}E_{\perp}(k) \tag{6.9}$$

where $E_{\parallel}$ and $E_{\perp}$ are the one-dimensional normalized energy spectra, one has

$$E_{\parallel}(k) = \frac{2}{\pi}\int_0^\infty dr\ f(r)\cos kr \quad \text{and}$$

$$E_{\perp}(k) = \frac{2}{\pi}\int_0^\infty dr\ g(r)\cos kr \quad . \tag{6.10}$$

Applying these definitions, one obtains

$$\overline{\left(\frac{\partial^n u}{\partial x^n}\right)^2} = \overline{u^2}\int_0^\infty dk\ E(k)\ k^{2n} = \overline{u^2}\cdot\overline{k^{2n}} \tag{6.11}$$

as a convenient short-hand expression for the scalar invariant considered. Substituting in the expansion (6.6), we obtain the equation relating the energy spectrum to the Eulerian autocorrelation for fast probing, i.e. for $v^2 \gg \overline{u^2}$,

$$R_E(\tau) = \int_0^\infty dk\ E(k)\ \cos(v\tau k)\ . \tag{6.12}$$

This autocorrelation has the same simple scaling properties (4.17) as the one-dimensional analogue considered in section 4.

We may now turn to the three-dimensional Lagrangian probing, neglecting the motion of the disturbance centres. The Taylor expansion of (6.2) leads to the problem of evaluating the variance of the derivatives

$$\left(\frac{d^n\vec{r}}{dt^n}\right)_L = (\vec{u}\cdot\vec{\nabla})(\vec{u}\cdot\vec{\nabla})\ldots(\vec{u}\cdot\vec{\nabla})\vec{u}(\vec{r}) = (\vec{u}(\vec{r})\cdot\vec{\nabla})^n\vec{u}(\vec{r})\ , \tag{6.13}$$

where the symbolic scalar operator $(\vec{u}\cdot\vec{\nabla})^n$ should be understood as equivalent to letting the operator $(\vec{u}\cdot\vec{\nabla})$ operate n consecutive times in the way indicated in (6.13). From the analysis of section 4 leading to equation (4.23) we know what the effect is of representing $\vec{u}$ as the sum

$$\vec{u}(\vec{r}) = \sum_{i=-\infty}^{+\infty} \vec{F}_i(\vec{r} - \vec{R}_i) \tag{6.14}$$

of a very large number of overlapping randomly distributed disturbances of the type $\vec{F}_i(\vec{r}-\vec{R}_i)$. What was learned from the expansion in (4.23) is that the dominant term of the variance of an expression like (6.13) may be obtained in the limit of large Reynolds numbers by letting all the differential operators operate on the last $\vec{u}$ only, neglecting the $\vec{r}$ dependence of the others, and furthermore in the same limit by considering the scalar operator $(\vec{u}\cdot\vec{\nabla})^n$ as statistically independent of the last $\vec{u}$ on which it operates. In symbolic form these steps may be expressed by

$$\overline{\left(\frac{d^n\vec{r}}{dt^n}\right)^2} = \overline{\left[(\vec{u}\cdot\vec{\nabla})^n\vec{u}(\vec{r})\right]^2} \simeq \overline{\left[|\vec{u}|^n(\vec{e}\cdot\vec{\nabla})^n\vec{u}(\vec{r})\right]^2} \simeq$$

$$\overline{|\vec{u}|^{2n}}\cdot\overline{\left[(\vec{e}\cdot\vec{\nabla})^n u\right]^2}\ , \tag{6.15}$$

where e is a unit vector, the direction of which is now independent of the direction of u. We thus obtain the relation corresponding to (6.8) above

$$\frac{1}{\overline{u^2}} \overline{\left(\frac{d^n \vec{r}}{dt^n}\right)^2}_L \simeq \overline{|\vec{u}|^{2n}} \; \frac{1}{\overline{u^2}} \overline{\left(\frac{\partial^n \vec{u}}{\partial x^n}\right)^2} = \overline{|\vec{u}|^{2n}} \cdot \overline{k^{2n}} \quad . \tag{6.16}$$

I may be observed that in the present limit the only difference between the Eulerian and the Lagrangian average is that whereas the first is proportional to v^{2n}, the second is proportional to $\overline{|\vec{u}|^{2n}}$, and the evaluation of the Lagrangian autocorrelation is thus reduced to the problem of evaluating moments of the size of the random velocity $\vec{u}$, which is known to have components distributed according to a Gaussian with variance $u'^2 = \frac{1}{3}\overline{u^2}$, where u' denotes the root mean square of any single velocity component.

The evaluation of $\overline{u^{2n}} = \overline{|\vec{u}|^{2n}}$ is easily performed when one remembers that the size of a three-dimensional vector is distributed according to a Maxwell distribution when each of its components has a Gaussian distribution. The distribution of u is thus given by the normalized Maxwell distribution function

$$P(u) = \sqrt{\frac{2}{\pi}} \; \frac{u^2}{u'^3} \; \exp\left[-\frac{u^2}{2u'^2}\right] \quad , \tag{6.17}$$

which leads to the values

$$\overline{u^{2n}} = \frac{(2n+1)!}{2^n n!} \; u'^{2n} \tag{6.18}$$

for the moments under consideration.

Inserting the Lagrangian average (6.16) into the expansion for the Lagrangian autocorrelation coefficient and using the above result (6.18), one obtains the following functional relation between the energy spectrum E(k) and the Lagrangian autocorrelation coefficient, in which disturbance centre motion has been neglected:

$$R_L(\tau) = \int_0^\infty dk \; E(k) \exp\left[-\frac{1}{2} u'^2 \tau^2 k^2\right] (1 - u'^2 \tau^2 k^2) =$$

$$\int_0^\infty dk \; E(k) \left[-\frac{d^2}{dz^2} e^{-\frac{z^2}{2}}\right]_{z=u'\tau k} \quad . \tag{6.19}$$

This expression is the three-dimensional analogue of equation (4.24). It is worth noting that, in contrast to the one-dimensional case, the function with which the energy spectrum should be folded in the three-dimensional case is not positive definite. Thus it is possible to obtain negative autocorrelations even for Lagrangian probing. This happens for instance if for theoretical reasons a δ-function spectrum is considered. However, the Lagrangian autocorrelation is always strongly damped with time even in this very special case, which corresponds to an extremely regular pattern of motion, as is apparent from the corresponding periodicity of the Eulerian autocorrelation.

We may now approach the slightly more complex problem of taking the motion of the disturbance centres into account also in the three-dimensional case. Using exactly the same kind of arguments as those leading to the equation (6.16), one obtains for the Lagrangian and Eulerian expansion coefficients the approximate expressions

$$\frac{1}{\overline{u^2}} \overline{\left(\frac{d^n \vec{r}}{dt^n}\right)_L^2} \simeq \overline{k^{2n}} \cdot \overline{|\vec{u}_1 - \vec{u}_2|^{2n}} \qquad (6.20)$$

and

$$\frac{1}{\overline{u^2}} \overline{\left(\frac{d^n \vec{u}}{dt^n}\right)_E^2} \simeq \overline{k^{2n}} \cdot \overline{|\vec{v} - \vec{u}|^{2n}} \quad ,$$

where $\vec{u}_1$ and $\vec{u}_2$ are two independent random velocities which in the Lagrangian probing have the same Gaussian distribution for each of their components, while $\vec{v}$ and $\vec{u}$ are the constant probing velocity and the random velocity of the fluid, respectively. Thus also this problem is reduced to the evaluation of moments of the size of relative velocities with known probability distributions for the two component velocities. This problem has been treated in another context by one of the authors (ref. 8) for the case of vectors the components of which are either constant or have a Gaussian distribution.

The result for the double-Gaussian case is that the moments are given by the expression

$$\overline{u_{12}^{2n}} = \overline{|\vec{u}_1 - \vec{u}_2|^{2n}} = \frac{(2n+1)!}{2^n n!} (u_1'^2 + u_2'^2)^n \quad , \qquad (6.22)$$

which in the case of equal distributions for $\vec{u}_1$ and $\vec{u}_2$ is reduced to the expression (6.18) with u' replaced by $\sqrt{2}u'$. Thus the Lagrangian auto-

correlation in the case where the motion of the disturbance centres is included has the form (6.19) with the above substitution of $2\,u'$ for u'.

One may, however, easily generalize the result to the case where the probe is moving with a random velocity the components of which have a Gaussian distribution different from that of the velocity of the fluid particles. This will for instance be the case when the probing is done with a balloon moving with a random velocity $\vec{u}_B$ which is obtained as some average of the velocity of the fluid over a volume of the size of the balloon and which is larger than the average volume of the disturbances contributing most of the fluctuating velocity $\vec{u}$ of the fluid particles. The corresponding autocorrelation function may then be obtained by inserting the expression (6.22) with $u_1' = u_B'$ and $u_2' = u'$ into the coefficient (6.20). The autocorrelation will then be related to the energy spectrum E(k) through the equation

$$R_B(\tau) = \int_0^\infty dk\ E(k) \left[-\frac{d^2}{dz^2}\ e^{-\frac{z^2}{2}} \right]_{z=\sqrt{u_B'^2 + u'^2}\,\tau k} \qquad (6.23)$$

It is instructive to consider two extreme cases of this equation, one corresponding to a very small "balloon" and the other to a very large one. The small balloon will faithfully follow the motion of the surrounding fluid, and u_B' may then be put equal to u'. In this limit the autocorrelation is of course reduced to the above-mentioned Lagrangian form. At the other extreme corresponding to a very large "balloon", the velocity $\vec{u}_B$ will be zero, and consequently we shall have $u_B' = 0$. This limit actually corresponds to the special case of Eulerian probing where the probing velocity $\vec{v}$ is zero. The autocorrelation function obtained in this limit happens to be identical with (6.19), which was the Lagrangian autocorrelation without motion of disturbance centres.

Turning finally to the case of Eulerian probing with arbitrary velocity $\vec{v}$, the inclusion of the motion of disturbance centres requires the evaluation of moments of the quantity $|\vec{v}-\vec{u}|$ in equation (6.21). The fact that we know the expression for the moments $\overline{|\vec{u}_1-\vec{u}_2|^{2n}}$ and the form of the Maxwell distribution function for $|\vec{u}_1|$ makes it possible to obtain the value of the moments $\overline{|\vec{v}-\vec{u}|^{2n}}$ by an inverse Laplace transformation (see ref. 8). The following result is obtained:

$$\overline{|\vec{v}-\vec{u}|^{2n}} = (-\frac{1}{2})^{n+1} \left[i \frac{\sqrt{2}\,u'}{v} H_{2n+1} (i \frac{v}{\sqrt{2}\,u'}) \right] u'^{2n} \qquad (6.24)$$

where H_{2n+1} are the Hermite polynomials obeying the equation

$$\sum_{m=0}^{\infty} (-1)^m H_{2m+1}(y) z^{2m+1}/(2m+1)! = \exp(z^2) \sin(2yz) \qquad (6.25)$$

This equation, which generates the Hermite polynomials of odd order, may actually be used to sum the Taylor expansion of the Eulerian autocorrelation function, and after some trivial mathematical manipulation the following relation is obtained between the energy spectrum E(k) and the Eulerian autocorrelation coefficient characterized by the constant probing velocity v and the r.m.s. of the fluctuating velocity u' :

$$R_E(\tau) = \int_0^{\infty} dk\, E(k) \exp(-\frac{u'^2\tau^2k^2}{2}) \left[\cos(v\tau k) - \frac{u'}{v}(u'\tau k) \sin(v\tau k) \right] =$$

$$\frac{u'}{v} \int_0^{\infty} dk\, E(k) \left\{ \frac{\partial}{\partial z} \left[\exp(-\frac{z^2}{2}) \sin(\frac{v}{u'} z) \right] \right\}_{z=u'\tau k} . \qquad (6.26)$$

As expected, this equation is reduced to the earlier derived equations (6.12) and (6.19) in the limits $v \gg u'$ and $v \ll u'$, respectively.

We have now carried through the programme of relating the energy spectrum E(k) in isotropic, homogeneous, well-developed turbulence to the different types of autocorrelation coefficients which may be obtained either by the Lagrangian probing, i.e. following a certain fluid particle in time, or by a "bolloon" probing with specified random Gaussian motion, or finally by a Eulerian probing specified by the value of the constant probing velocity v. In the last section we shall investigate the relation between the energy spectrum E(k) and the power spectra $P(\omega)$ obtained by the different probing methods mentioned above. Finally, we shall discuss how the Hay-Pasquill conjecture may be derived from the above relations as an approximation, and how the Sutton theory for turbulent diffusion is related to this theory.

7. Applications and Discussion

Having derived in the previous section the autocorrelation coefficients for the various ways of probing the velocity field, we are now in a position to relate the corresponding power spectra $P(\omega)$ which would be measured in these probings to the energy spectrum characterizing the turbulence. The relations are obtained by substitution of the respective autocorrelation coefficients in the general formula (1.4). In this way one obtains Lagrangian, "balloon" and Eulerian power spectra in terms of integral transforms of the energy spectrum E(k), using kernel functions characteristic of the way of probing.

In the case of Lagrangian and "balloon" probing the autocorrelation coefficient (6.23) is used, yielding the expression

$$P_{U'}(\omega) = \frac{1}{U'} \int_0^\infty \frac{dk}{k} E(k) K_L\left(\frac{\omega}{U'k}\right) , \qquad (7.1)$$

where ω is the frequency and U' is defined according to whether the probing is Lagrangian or "balloon" as

$$U' = \sqrt{2}\, u' \qquad \text{(Lagrangian)} \qquad (7.2)$$

or

$$U' = \sqrt{u_B'^2 + u'^2} \qquad \text{(balloon)}$$

while $K_L(z)$ is the normalized Lagrangian kernel function

$$K_L(z) = \sqrt{\frac{2}{\pi}}\, z^2 \exp\left(-\frac{z^2}{2}\right) . \qquad (7.3)$$

This function has a "universal" shape, which is shown in fig. 1. This would be the shape of the power spectrum in the extreme and unrealistic case in which the energy spectrum contained only one wave number k_o, i.e. in which $E(k) = \delta(k-k_o)$. Thus in these types of probing a particular wave number will never be associated with a single frequency, but rather with a frequency band with an average frequency and standard deviation given by

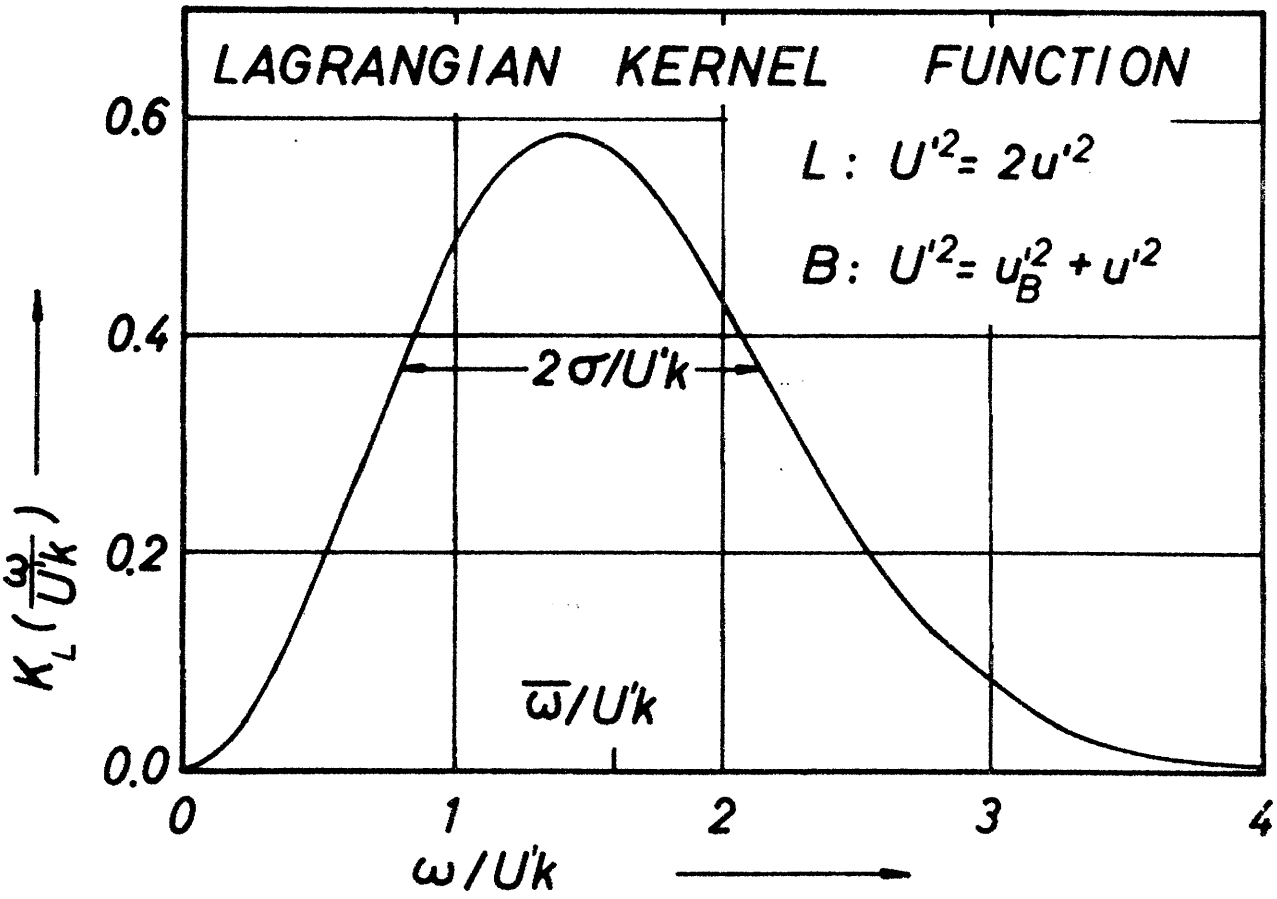

Fig. 1. The normalized Lagrangian kernel function K_L is shown as a function of $\omega/U'k$. Mean value and standard deviation of ω are given by $\bar{\omega}/U'k = \sqrt{8/\pi} \simeq 1.60$ and $\sigma/U'k \simeq 0.67$. Case L corresponds to Lagrangian probing. Case B corresponds to probing with a "balloon" moving with a random velocity $\vec{u}_B$ with components distributed according to a Gaussian with rms u'_B.

$$\bar{\omega} = \sqrt{\frac{8}{\pi}}\, U'k_o \simeq 1.60\, U'k_o$$

and (7.4)

$$\sigma = \sqrt{\overline{\omega^2} - \bar{\omega}^2} \simeq 0.67\, U'k_o \quad .$$

The general expression for the n'th-order moment of ω is

$$\overline{\omega^n} = (U'k_o)^n \frac{1}{\sqrt{\pi}}\, 2^{\frac{n+2}{2}}\, \Gamma\left(\frac{n+3}{2}\right) \quad . \tag{7.5}$$

Since most of the energy spectra occurring in nature have a rather smooth shape ranging over several decades in the wave number, the Lagrangian kernel may for many practical purposes be approximated by a δ-function yielding a one to one connection between the frequency and the wave number. It is then natural to choose the upper relation (7.4) to give this connection. In this approximation one obtains the following rough but simple relation between the Lagrangian power spectrum and the energy spectrum:

$$P_L(\omega) \simeq \frac{\sqrt{\pi}}{4}\, \frac{1}{u'}\, E\left(\frac{\sqrt{\pi}}{4}\, \frac{\omega}{u'}\right) . \tag{7.6}$$

In the case of Eulerian probing one proceeds in an analogous way by substituting the Eulerian autocorrelation coefficient (6.26) in the relation (1.4), obtaining the Eulerian power spectrum in the form

$$P_E(\omega) = \frac{1}{v} \int_0^\infty \frac{dk}{k}\, E(k)\, K_E\left(\frac{\omega}{vk}\, ,\, \frac{v}{\sqrt{2}\,u'}\right) , \tag{7.7}$$

where the normalized Eulerian kernel function is given by

$$K_E(z, y) = \frac{yz}{\sqrt{\pi}} \left\{ \exp\left[-y^2(z-1)^2\right] - \exp\left[-y^2(z+1)^2\right] \right\} \quad . \tag{7.8}$$

It is seen that this kernel contains a parameter which in (7.7) has the value $y = v/\sqrt{2}\,u'$. The shape of the kernel thus depends on the ratio between the probing velocity and the r.m.s. of the random part of the velocity. In fig. 2 the Eulerian kernel function is given for the three values 0.1, 1 and 10 of the parameter $y = v\sqrt{2}\,u'$. In the limit of small probing velocities ($v \ll u'$) the shape is similar to the one obtained for the Lagrangian kernel. This corresponds to the fact that in this limit Eulerian probing and "balloon"

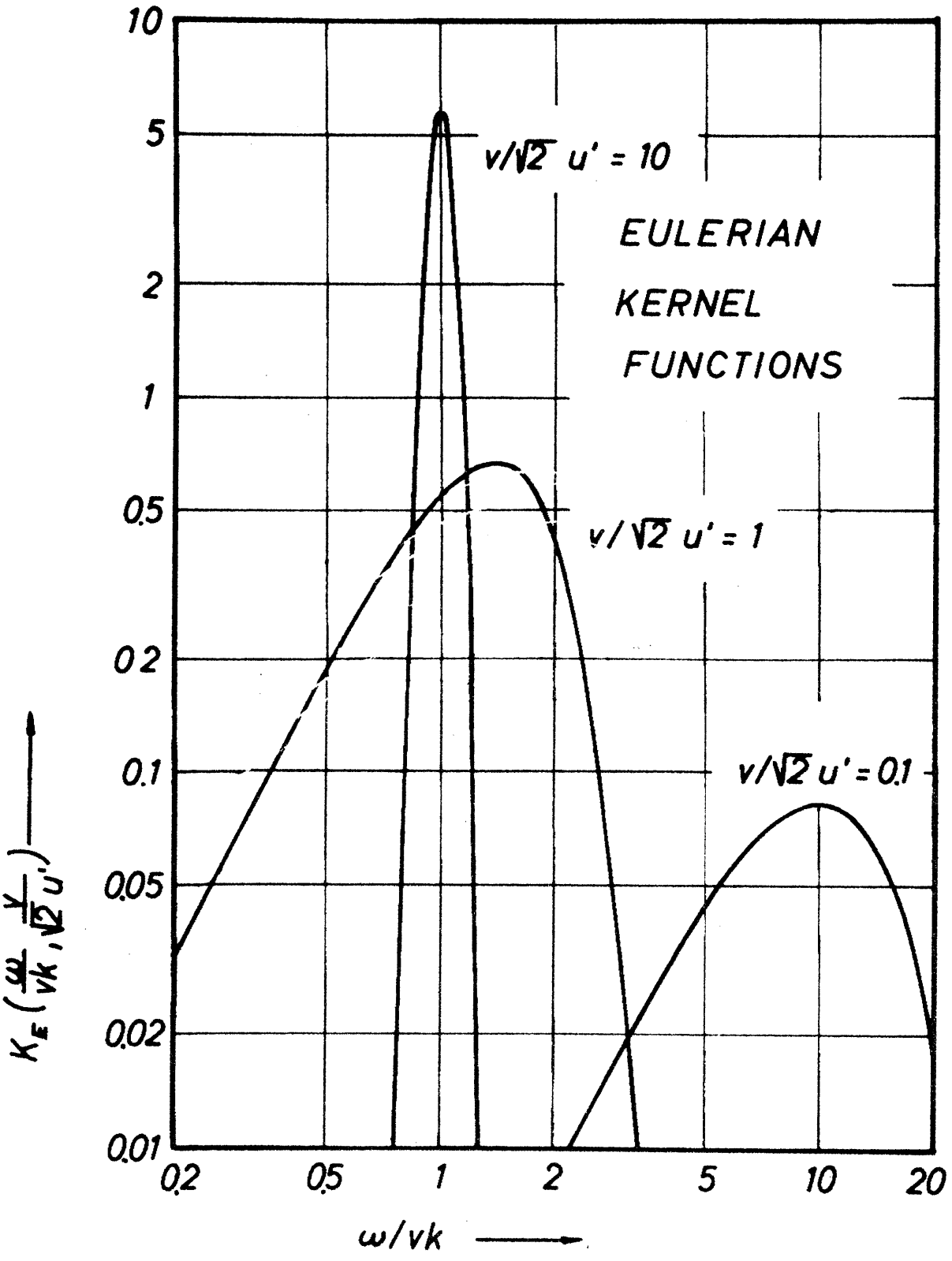

Fig. 2. Examples of normalized Eulerian kernel functions K_E for various ratios of the probing velocity v to the rms u' of a single component of the fluctuating part of the velocity field.

probing with $u'_B = 0$ are identical. In the opposite limit ($v \gg u'$) one also obtains a simple result since the kernel function then degenerates to a δ-function yielding the following simple one to one correspondence between wave numbers and frequencies:

$$\omega \simeq vk \quad . \tag{7.9}$$

This relation, which corresponds to the so-called Taylor hypothesis, connects the Eulerian power spectrum and the energy spectrum by the simple scaling law

$$P_E(\omega) \simeq \frac{1}{v} E(\frac{\omega}{v}) \quad . \tag{7.10}$$

The general trend of the Eulerian kernel functions is indicated in fig. 3, where the average frequency $\bar{\omega}$ and the standard deviation σ corresponding to a particular wave number k are plotted in dimensionless units as functions of the parameter $v/\sqrt{2}\,u'$. For completeness the general expression for the n'th moment of ω corresponding to the wave number k is given below:

$$\overline{\omega^n} = (vk)^n \frac{\Gamma(n+2)}{\sqrt{2\pi}} (\frac{u'}{v})^{n+1} \exp\left[- \frac{v^2}{4u'^2} \right] \times$$

$$\left\{ D_{-n-2}\left[- \frac{v}{u'} \right] - D_{-n-2}\left[\frac{v}{u'} \right] \right\} \quad , \tag{7.11}$$

where D is the parabolic cylinder function of negative order defined by (see ref. 6)

$$D_{-m-1}(z) = \sqrt{2}\, \frac{(-1)^m}{m!} \exp(- \frac{z^2}{4}) \frac{d^m}{dz^m} \left[\exp \frac{z^2}{2} \operatorname{Erfc}(\frac{z}{\sqrt{2}}) \right]. \tag{7.12}$$

It is quite clear from fig. 3 how the transformation relation (7.9) so frequently used for instance in wind tunnel experiments becomes a good approximation for $v \gg u'$ and actually for many purposes may be sufficient for v a few times larger than u' if a not too detailed knowledge of the energy spectrum E(k) is needed.

It is now also obvious how the relation (1.5) conjectured by Hay and Pasquill may be obtained as a first rough approximation in the case of rather smooth energy spectra E(k). By elimination of the energy spectrum, using the relations (7.6) and (7.10), one obtains the following asymptotic relation between the two types of power spectra:

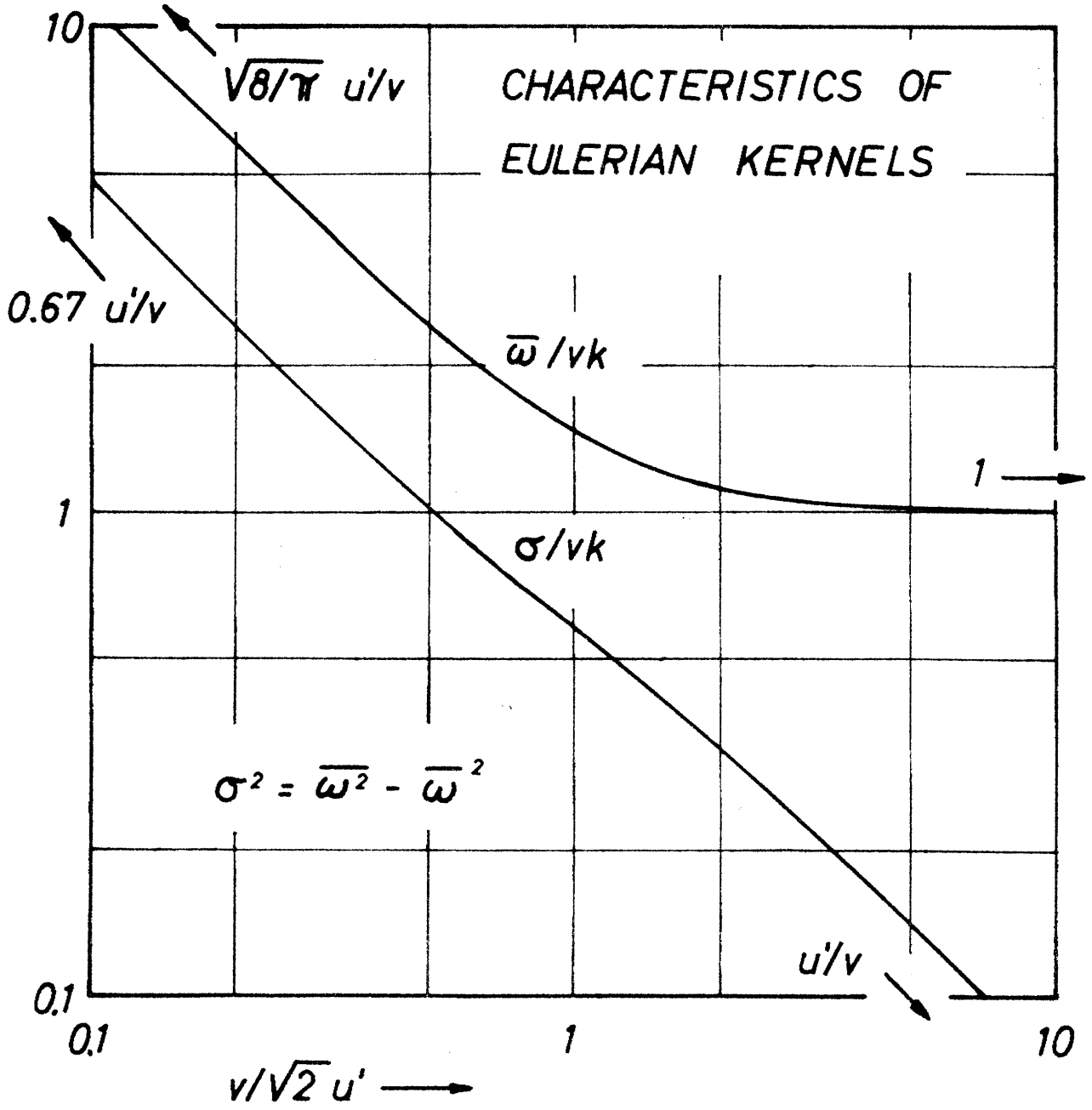

Fig. 3. Mean value and standard deviation of ω in units of vk as functions of the parameter $v/\sqrt{2}\,u'$ for the normalized Eulerian kernel function K_E.

$$P_L(\omega) \simeq \beta P_E(\beta \omega) \ , \tag{7.13}$$

where

$$\beta = \frac{\sqrt{\pi}}{4} \frac{v}{u'} \tag{7.14}$$

in the limit where $v \gg u'$. The relation obviously breaks down when v becomes of the same magnitude as u' since, as mentioned above, Eulerian probing with a velocity $v \ll u'$ corresponds to "balloon" probing with $u'_B = 0$. Thus a lower limit for β is obtained by comparing the Lagrangian power spectrum with this particular type of "balloon" spectrum. The comparison yields the value

$$\beta = \frac{1}{\sqrt{2}} \tag{7.15}$$

in the limit where $v \ll u'$.

It is of interest to apply this theory to the phenomenon of turbulent diffusion. In order to investigate the contribution of a narrow wave number range of the energy spectrum E(k) to the diffusion process one may insert in the Taylor relation (1.1) the Lagrangian autocorrelation coefficient corresponding to an energy spectrum of the singular shape $E(k) = \delta(k-k_o)$. After performance of the double integration over time the following variance is obtained for the displacement r along any one of the three independent space directions:

$$r'^2(t) = \frac{1}{k_o^2} \left[1 - \exp(u'^2 k_o^2 t^2) \right] . \tag{7.16}$$

Thus the probability distribution for finding a displacement r in a prescribed direction converges toward a limiting distribution with the variance k_o^{-2} in a time which is large compared with the period $\tau_o = (u'k_o)^{-1}$. At times small compared with τ_o the standard deviation r' grows with time approximately as u't.

The spectra occurring in natural turbulence usually have a smooth shape, and the diffusion observed is the co-operative effect of all parts of the spectrum. However, the above consideration shows how the high wave number part of the spectrum is gradually rendered inactive as the diffusion proceeds while only that part of the spectrum which has wave numbers

$k \ll (u't)^{-1}$ is important for further diffusion. It is this progressive reduction of the part of the energy spectrum which is active in the diffusion process which results for a wide class of spectra in the well-known Fick law of turbulent diffusion. In order to illustrate this statement one may consider some very simple spectra for which the calculation may be carried out in terms of elementary functions.

Let us first consider a normalized spectrum

$$E(k) = \frac{2\lambda}{\sqrt{\pi}} \exp(-\lambda^2 k^2) \tag{7.17}$$

characterized by just one parameter λ. The resulting expression for the variance with its asymptotic values for small and large values of t is

$$r'^2 = 2\lambda^2 \left[(1 + \frac{u'^2}{\lambda^2} t^2)^{1/2} - 1 \right] \simeq \begin{cases} u'^2 t^2 & \text{for } t \ll \frac{\lambda}{u'} \\ 2u'\lambda t & \text{for } t \gg \frac{\lambda}{u'} \end{cases} . \tag{7.18}$$

Since this spectrum is generally associated with the last stages of decaying turbulence, where one cannot expect the approximations underlying the present statistical theory to apply, one might also consider a slightly more realistic spectrum which has a qualitative resemblance to the ones expected in fully developed turbulence. This is true of the normalized spectrum

$$E(k) = \frac{2}{\pi} \frac{\lambda}{1 + \lambda^2 k^2} , \tag{7.19}$$

which yields the following approximate expression for the variance of the displacement after a time t:

$$r'^2 \simeq \lambda^2 \frac{u't/\lambda}{1 + \lambda/(u't)} \simeq \begin{cases} u'^2 t^2 & \text{for } t \ll \frac{\lambda}{u'} \\ u'\lambda t & \text{for } t \gg \frac{\lambda}{u'} \end{cases} . \tag{7.20}$$

Thus, appart from numerical constants, the same general behaviour is obtained.

For more general spectral shapes the following expression may be obtained by substitution of the general expression for the Lagrangian auto-correlation coefficient in the relation (1.1):

$$r'^2(t) = D(t) \cdot t , \tag{7.21}$$

where the time-dependent diffusion parameter is given by an integral transformation of the energy spectrum E(k) through

$$D(t) = u' \int_0^\infty \frac{dk}{k} E(k)\ T(u'tk) \quad , \tag{7.22}$$

in which the kernel function T has the explicit universal form

$$T(z) = \frac{1 - e^{-z^2}}{z} \quad . \tag{7.23}$$

For many practical purposes and in order to get an intuitive understanding of the turbulent diffusion mechanism, the kernel function T may be approximated by the somewhat simpler function

$$T(z) \simeq e^{-|\ln z|} \quad . \tag{7.24}$$

This form and the fact that the effective part of the energy spectrum usually ranges over several decades in the wave number k suggest that a transformation should be made to a logarithmic scale in the wave number. If this is done by introduction of the parameter $s = -\ln k$ in the expression (7.22), one obtains for D(t) the expression

$$D(t) = u' \int_{-\infty}^{\infty} ds\ E(e^{-s})\ T(e^{-s+\ln(u't)}) \quad . \tag{7.25}$$

In fig. 4 the function T and a simple arbitrary energy spectrum E have been drawn on a logarithmic wave number scale. The instantaneous value of the diffusion parameter D is then obtained by integration of the product of the two functions over the entire range of s. The time dependence of D comes about as the kernel function T advances without change of shape toward larger wave numbers with an abscissa proportional to lnt. In the initial stages this gives rise to enhanced diffusion by making the "overlap" integral in D proportional to t. At later times, when passing into the region where E(k) is flat, D(t) becomes almost constant, and Fickian diffusion is obtained. This behaviour thus covers the range of Sutton's diffusion formula relating it directly to the shape of the energy spectrum. It must, however, be remembered that the present theory is restricted to the case of isotropic turbulence and thus gives rise to isotropic

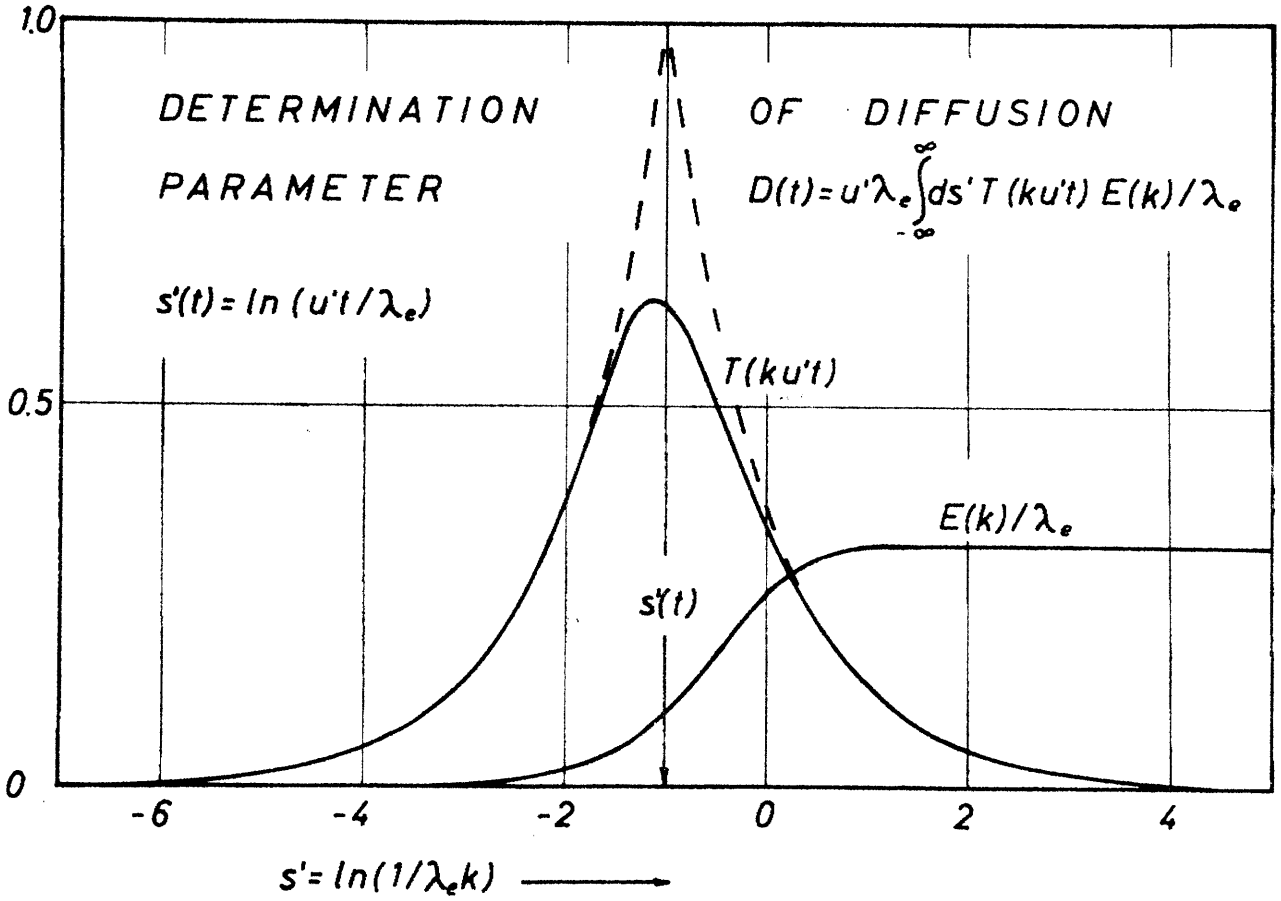

Fig. 4. Diagram showing the functions used for the determination of the time-dependent diffusion parameter $D(t) = u' \int_{-\infty}^{\infty} ds\ E(k)T(u'tk)$ with $k = \exp(-s)$. The function T moves without change of shape toward larger s values with a speed proportional to lnt. The transformation $s' = s - \ln \lambda_e$ is introduced for scaling purposes. The dashed lines correspond to the approximation (7.24).

diffusion. Furthermore it is inherent in the approximations made that fully developed turbulence is presupposed over the wave number range considered since the statistical assumptions imply a large amount of overlap of the disturbances. Thus, in situations where the components of the random part of the velocity field have a non-Gaussian distribution, the application of the present theory would be expected to have a limited validity or might even be misleading. Further applications and extensions of the present theory to special cases of turbulent diffusion are discussed by one of us in ref. 9.

Appendix

In this appendix we shall extend and generalize some of the theorems, proved by Rice (ref. 7), about random processes of the shot effect type.

Let us consider a random process which may be expressed as a sum

$$u(x) = \sum_{i=-\infty}^{+\infty} F_i(x-X_i) \quad , \tag{A.1}$$

where each term F_i represents a disturbance statistically independent of the others with a random shape and a random position X_i. More precisely, this means that the shapes of the disturbances may be classified into a possibly infinite number of types for which there exists a known probability distribution. The random distribution of the positions X_i is characterized by a density μ so that, within any interval X which is large compared with the range of a single disturbance, the number N of disturbances occurring in this interval is distributed according to a Poisson distribution with a mean value $\overline{N} = \mu X$, while the position X_i of each of these N signals will have the probability dx/X of occurring in the subinterval dx within X.

Taking derivatives with respect to x in equation (A.1), one obviously obtains the random processes

$$\frac{d^m u}{dx^m} = \sum_{i=-\infty}^{+\infty} \frac{\partial^m F_i}{\partial x^m} \qquad (m=0,1,\ldots,n) \quad , \tag{A.2}$$

of which (A.1) is nothing but the special case m = 0.

Our aim is to calculate expectation values of products of u and its derivatives, each raised to arbitrary powers p_m, hence having the general form

$$\overline{\prod_{m=o}^{n} \left(\frac{d^m u}{dx^m}\right)^{p_m}} \quad , \tag{A.3}$$

and to express them in terms of the irreducible mixed moments of a single disturbance and its derivatives,

$$\left\langle \int_{-\infty}^{+\infty} dx \prod_{m=o}^{n} \left(\frac{\partial^m F_i}{\partial x^m} \right)^{\pi_m} \right\rangle_i , \qquad (A.4)$$

where $\langle \ \rangle_i$ indicates an averaging over the disturbance shapes.

To establish this relation we shall use the method of characteristic functions (ch. f.).

In order to be able to neglect end effects, we shall perform all averages over an interval X which is long compared with the range of the disturbances. Furthermore, we shall first treat the special case where $u = u_1(x)$ has just one disturbance F_i in the interval X. In this case the joint ch. f. for u_1 and its derivatives up to and including the n'th is given by

$$\Phi_1(\varkappa_o, \ldots, \varkappa_n) = \overline{\exp\left[i \sum_{m=o}^{n} \varkappa_m \frac{\partial^m F_i}{\partial x^m} \right]} =$$

$$\left\langle \frac{1}{X} \int_{-X/2}^{+X/2} dX_i \ \exp\left[i \sum_{m=o}^{n} \varkappa_m \frac{\partial^m F_i}{\partial x^m} \right] \right\rangle_i . \qquad (A.5)$$

If it is assumed, as done in section 3 above, that the probability distribution of shapes has the special property that shapes $F_i(x)$ and $-F_i(x)$ are equally probable, the ch. f. (A.5) will be real valued and an even function in all $\varkappa_m$.

Recalling that moments of u_1 and its derivatives may be obtained by differentiation of the ch. f., we obtain

$$\overline{\prod_{m=o}^{n} \left(\frac{d^m u_1}{dx^m} \right)^{\pi_m}} = \left\langle \frac{1}{X} \int_{-X/2}^{+X/2} dX_i \prod_{m=o}^{n} \left(\frac{\partial^m F_i}{\partial x^m} \right)^m \right\rangle_i = \qquad (A.6)$$

$$\left[\prod_{m=o}^{n} \left(\frac{1}{i} \frac{\partial}{\partial \varkappa_m} \right)^{\pi_m} \Phi_1 \right]_{\varkappa_o = \ldots = \varkappa_n = 0} .$$

Under the above special assumption about the symmetry of the shape distribution we immediately see that the moment (A.6) vanishes if

if $\pi = \sum_{m}^{n} \pi_m$ is an odd integer. This result will be used later, but is not essential for the following considerations, which also apply to more general shape probability distributions.

In the case where exactly N disturbances appear in the interval X, the n+1 random processes we consider have the form

$$\frac{d^m u_N}{dx^m} = \sum_{i=1}^{N} \frac{\partial^m F_i}{\partial x^m} \qquad (m = 0, 1, \ldots, n) \qquad (A.7)$$

in the interval under consideration.

We may now construct the corresponding joint ch. f. in the usual way and obtain

$$\Phi_N(\varkappa_o, \ldots, \varkappa_n) = \overline{\exp\left[i \sum_{m=o}^{n} \varkappa_m \left(\sum_{i=1}^{N} \frac{\partial^m F_i}{\partial x^m} \right) \right]} = \Phi_1^N \qquad (A.8)$$

since all N disturbances are assumed statistically independent.

In the general case the number of disturbances N appearing in the interval X is assumed to be distributed according to a Poisson distribution of the form

$$P_{\overline{N}}(N) = \frac{\overline{N}^N \exp(-\overline{N})}{N!} \qquad (A.9)$$

with $\overline{N} = \mu X$. Thus we obtain the joint ch. f. corresponding to the random processes defined in (A. 2) by forming the expectation value of (A. 8) with the distribution (A. 9):

$$\Phi(\varkappa_o, \ldots, \varkappa_n) = \sum_{N=o}^{\infty} P_{\overline{N}}(N)\, \Phi_N = \exp\left[\overline{N}(\Phi_1 - 1) \right] . \qquad (A.10)$$

In principle we have now solved the problem of evaluating moments of the type given in (A. 3) since they may be obtained by differentiation of Φ through the relation

$$\overline{\prod_{m=o}^{n} \left(\frac{d^m u}{dx^m}\right)^{p_m}} = \left[\prod_{m=o}^{n} \left(\frac{1}{i}\frac{\partial}{\partial \varkappa_m}\right)^{p_m} \Phi\right]_{\varkappa_o = \ldots = \varkappa_n = o} . \qquad (A.11)$$

In order to obtain the dominant terms of these moments for large values of μ we shall, however, expand the ch. f. Φ given by (A.10) in terms of $\overline{N} = \mu X$ since this is the only quantity containing μ. The large interval X used for the purpose of averaging must of course eventually disappear from our equations since it was chosen arbitrarily.

Expanding the exponential function in equation (A.10), we obtain

$$\Phi = \sum_{s=o}^{\infty} \overline{N}^s \frac{(\Phi_1 - 1)^s}{s!} . \qquad (A.12)$$

The differentiations indicated in (A.11) may then be performed separately on each term of the expansion (A.12). For that purpose we shall, however, need an extension of Leibnitz' rule for the differentiation of a product of two variables, to the case of a product of s variables.

Let $y_1(\varkappa_o, \ldots, \varkappa_n), \ldots, y_s(\varkappa_o, \ldots, \varkappa_n)$ be a set of s different functions of the n+1 variables $\varkappa_o, \ldots, \varkappa_n$. On application of the multinominal theorem instead of the binominal, Leibnitz' formula for the derivatives of a product of two functions is easily generalized to the case of s functions, yielding for the p_o'th derivative with respect to $\varkappa_o$

$$\frac{1}{p_o!}\left(\frac{\partial}{\partial \varkappa_o}\right)^{p_o} \prod_{k=1}^{s} y_k = \sum_{\substack{s \\ \sum_{k=1} \pi_{o,k} = p_o}} \left[\prod_{k=1}^{s} \frac{1}{\pi_{o,k}!}\left(\frac{\partial}{\partial \varkappa_o}\right)^{\pi_{o,k}} y_k\right] , \qquad (A.13)$$

where the summation is to be extended over all different sets $(\pi_{o,1}, \ldots, \pi_{o,s})$ which satisfy the relation $\sum_{k=1}^{s} \pi_{o,k} = p_o$. By repeated differentiation with respect to the other variables, the following generalized expression is obtained:

$$\left[\prod_{m=o}^{n} \frac{1}{p_m!}\left(\frac{\partial}{\partial \varkappa_m}\right)^{p_m}\right]\left[\prod_{k=1}^{s} y_k\right] = \sum_{\substack{\sum_{k=i}^{s} \pi_{m,k}=p_m \\ m=o,\ldots,n}} \left\{\prod_{k=1}^{s}\left[\prod_{m=o}^{n} \frac{1}{\pi_{m,k}!}\left(\frac{\partial}{\partial \varkappa_m}\right)^{\pi_{m,k}} y_k\right]\right\}. \tag{A.14}$$

Since we shall only need derivatives of powers of a single dependent variable, we identify all the functions y_k in (A.14) by a single function $y(\varkappa_o, \ldots, \varkappa_n)$. Simple combinatorial reasoning then leads to the following specialized version of equation (A.14):

$$\left[\prod_{m=o}^{n} \frac{1}{p_m!}\left(\frac{\partial}{\partial \varkappa_m}\right)^{p_m}\right]\left[\frac{y^s}{s!}\right] =$$

$$\sum_{\text{sol}}^{(p_o,\ldots,p_n)} \prod_{(\pi_o,\ldots,\pi_n)=(o,\ldots,o)} \frac{1}{q_{\pi_o,\ldots,\pi_n}!} \left\{\left[\prod_{m=o}^{n} \frac{1}{\pi_m!}\left(\frac{\partial}{\partial \varkappa_m}\right)^{\pi_m}\right] y\right\}^{q_{\pi_o,\ldots,\pi_n}}. \tag{A.15}$$

The summation should be extended over all possible different non-negative integer solutions to the Diophantine equations (A.16) and (A.17)

$$\sum_{(\pi_o,\ldots,\pi_n)=(o,\ldots,o)}^{(p_o,\ldots,p_n)} q_{\pi_o,\ldots,\pi_n} = s \tag{A.16}$$

$$\sum_{(\pi_o,\ldots,\pi_m)=(o,\ldots,o)}^{(p_o,\ldots,p_n)} \pi_m \; q_{\pi_o,\ldots,\pi_m,\ldots,\pi_n} = p_m \quad (m=0,1,\ldots,n) \; . \tag{A.17}$$

The product over $(\pi_o, \ldots, \pi_n)$ in (A.15) and the corresponding sums in (A.16) and (A.17) are understood to run over <u>all</u> different sets $(\pi_o, \ldots, \pi_n)$ with the only restriction that $o \leq \pi_m \leq p_m$ for $m = 0, 1, \ldots, n$.

In order to calculate the mixed moments (A. 3) we may now insert the expansion (A. 12) for the joint ch. f. into the expression (A. 11). On application of the relation (A. 15), derived above, to each term in the expansion, the following equation is obtained after a slight rearrangement:

$$\overline{\prod_{m=o}^{n} \frac{1}{p_m!} \left(\frac{d^m u}{dx^m}\right)^{p_m}} = \sum_{s=o}^{\infty} \overline{N}^s \left\{ \sum_{sol\ q} \prod_{(o,\ldots,o)}^{(p_o,\ldots,p_n)} \frac{1}{q_{\pi_o,\ldots,\pi_n}!} \times \right.$$

$$\left. \left[\prod_{m=o}^{n} \frac{1}{\pi_m!} \left(\frac{1}{i} \frac{\partial}{\partial \varkappa_m}\right)^{\pi_m} (\Phi_1 - 1) \right]_{\varkappa_o = \ldots = \varkappa_n = o}^{q_{\pi_o,\ldots,\pi_n}} \right\} \quad ; \qquad (A.18)$$

here the summation marked "sol q" is again to be performed over all solutions to equations (A. 16) and (A. 17).

From equation (A. 18) and the conditions (A. 16) and (A. 17) it is possible to derive the consequences listed below.

(i) Since $\Phi_1 - 1$ is equal to zero for $\varkappa_o = \varkappa_1 = \ldots \varkappa_n = o$, only such solutions to equations (A. 16) and (A. 17) for which we have

$$q_{\pi_o,\ldots,\pi_n} = o \qquad \text{for} \qquad \pi_o = \ldots = \pi_n = o \qquad (A.19)$$

can give non-zero contributions to the sum over "sol q" in (A. 18). This means that we may add the condition (A. 19) to the restricting equations on the q's without affecting the right side of equation (A. 18).

(ii) The sum over s in equation (A. 18) is a finite sum since we may establish an upper bound for the values of s which will give non-zero contributions to the sum over solutions to the restricting equations. This may be seen by summing the n+1 equations (A. 17); with the help of (A. 16) and (A. 19) one obtains

$$\sum_{m=o}^{n} p_m = \sum_{\substack{(p_o,\ldots,p_n) \\ (\pi_o,\ldots,\pi_n) = \\ (o,\ldots,o)}} \left[\sum_{m=o}^{n} \pi_m q_{\pi_o,\ldots,\pi_m,\ldots,\pi_n} \right] \geq$$

$$\sum_{\substack{(p_o,\ldots,p_n) \\ (\pi_o,\ldots,\pi_n) = \\ (o,\ldots,o)}} q_{\pi_o,\ldots,\pi_n} = s \quad . \tag{A. 20}$$

The validity of the inequality is realized by observing that because of the restriction (A. 19) there is no contribution to the last sum from the term $q_{o,o,\ldots,o}$; thus the preceding sum is term by term larger than or equal to the last. Therefore $p = \sum_{m=o}^{n} p_m$ is an upper bound for the exponent of highest order s_{max} in the expansion (A. 18).

(iii) If the shape distribution is symmetric, as described above in connection with equation (A. 6), all terms for which $\pi = \sum_{m=o}^{n} \pi_m$ is an odd integer vanish, and we may thus under this assumption for the distribution of shapes impose a further condition on the q's without changing the value of the right side of (A. 18). This condition is

$$q_{\pi_o,\ldots,\pi_n} = o \qquad \text{for} \qquad \pi = \sum_{m=o}^{n} \pi_m \text{ odd} \; . \tag{A. 21}$$

Application of this condition together with the first equality in (A. 20) leads to the conclusion that all moments (A. 3) for which $p = \sum_{m=o}^{n} p_m$ is an odd integer will vanish. Thus the symmetry assumption eliminates all mixed moments of odd order, as might also be proved in a more direct way by a simple parity argument.

(iv) In general the largest values of s are obtained when the non-vanishing q's have the least possible index sum $\pi = \sum_{m=o}^{n} \pi_m$. Since the conditions (A. 19) and (A. 21) exclude the possibility of non-vanishing q's for π equal to 0 and 1, the largest s in the case of symmetric shape distributions is obtained when all q's vanish except when π is equal to 2, that is, when according to the relation (A. 6) only pair correlations appear in the

corresponding term. As seen from (A. 6), the pair correlations may be either quadratic or cross term according to whether the q's are of the type $q_{o,\ldots,o,2,o,\ldots,o}$ or $q_{o,\ldots,o,1,o,\ldots,o,1,o,\ldots,o}$. Under these circumstances the maximum value of s as determined from the inequality (A. 20) is $s_{max} = p/2$.

Finally, we shall rewrite equation (A. 18) in the form (A. 22), thus introducing the irreducible mixed moments (A. 4), by applying equation (A. 6). Elimination of the dummy range X is accomplished partly by cancellation and partly by extension of the integration limits to infinity; this extension is justified since X was chosen large compared with the range of the disturbances considered.

$$\overline{\prod_{m=o}^{n} \frac{1}{p_m!}\left(\frac{d^m u}{dx^m}\right)^{p_m}} = \sum_{s=o}^{s_{max}} \mu^s \left\{ \sum_{sol\ q(\pi_o,\ldots,\pi_n)=(o,\ldots,o)}^{(p_o,\ldots,p_n)} \prod \frac{1}{q_{\pi_o,\ldots,\pi_n}!} \times \right.$$

$$\left. \left\langle \int_{-\infty}^{+\infty} dx \prod_{m=o}^{n} \frac{1}{\pi_m!}\left(\frac{\partial^m F_i}{\partial x^m}\right)^{\pi_m} \right\rangle_i^{q_{\pi_o,\ldots,\pi_n}} \right\} . \qquad (A. 22)$$

This equation has the form we set out to find since it gives an explicit expansion of the mixed moments of u and its derivatives in terms of the disturbance density μ with coefficients containing only the irreducible mixed moments (A. 4).

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